"A STUDY OF FIXED POINT THEOREMS IN TOPOLOGICAL SPACE"

Dissertation submitted to the Department of Mathematics in partial fulfillment of the requirements for the award of the degree of Master of Science in Mathematics

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Certificate

This is to certify that the dissertation entitled " A STUDY OF FIXED POINT THEOREMS IN TOPOLOGICAL SPACE "sub- mitted by Mustak Ahmed Roll No. MAT-11/23, Registration No. MSSV-0023-101- 001330, in partial fulfillment for M.Sc in Mathematics, is a bonafide record of original work carried out under my supervision and guidance.

To the best of my knowledge, the work has not been submitted earlier to any other institution for the award of any degree or diploma.

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DECLARATION

I, Mustak Ahmed, hereby declare that the dissertation titled "A STUDY OF FIXED-POINT THEOREMS IN TOPOLOGICAL SPACE "submitted to the Department of Mathematics, Mahapurusha Srimanta Sankaradeva Viswavidyalaya, is a record of original work carried out by me under the supervision of Dr. Raju Bordoloi, HOD.

This work has not been submitted earlier to any other institution or university for the award of any degree or diploma.

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Chapter 1

Introduction

1.1 Background

In recent years, mathematics has continued to evolve, branching into new areas and tackling increasingly complex problems. Our research fits into this dynamic landscape by focusing on a particularly intricate and promising domain: the study of complex-valued integro-differential and integral equations, especially within controlled metric spaces.

While these equations are recognized in the current literature, there's still a real need for more advanced tools—particularly those that can handle the uncertainty that often accompanies real-world problems. That's where our work comes in.

We're exploring this challenge through an innovative combination of approaches. First, we propose a new extension of the well-known Fisher and Banach contraction theorems. To complement this, we apply the fractional Adams-Bashforth method, enhancing our ability to identify common solutions in uncertain settings. Together, these techniques offer a fresh perspective and practical tools for addressing complex mathematical systems.

A cornerstone of this exploration is fixed point theory, which remains a vital area in mathematics due to its deep connections with analysis, topology, and geometry. It has far-reaching applications—from solving systems of linear equations and analyzing multivalued mappings, to addressing ordinary and partial differential equations, and tackling Fredholm and Volterra-type integral equations. The versatility of fixed-point theory is only growing, and there's still a wide horizon of discoveries waiting to be made.

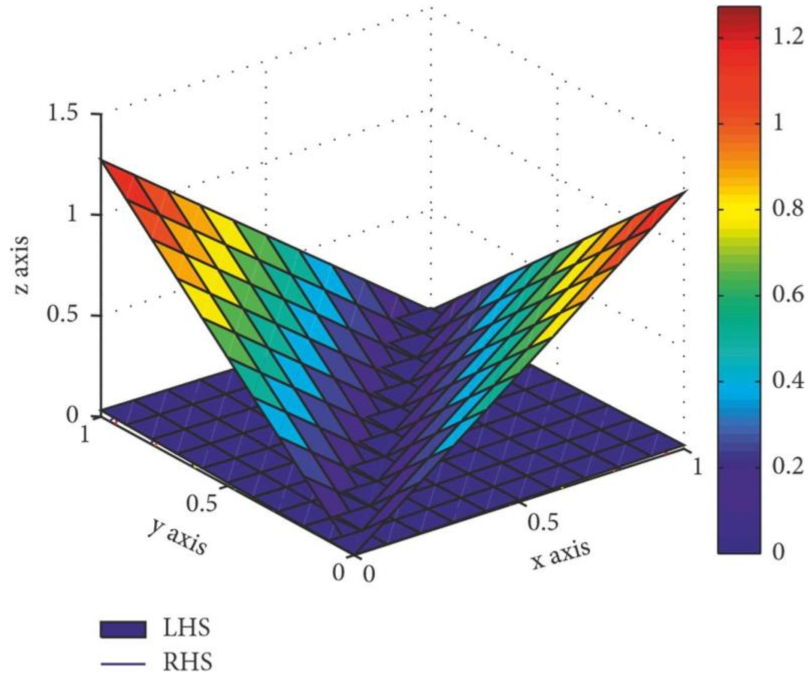

Figure 1.1: Unique Solution of Urysohn and Fredholm Integral Equations Fixed point Solution

In the 19th century, Banach introduced a groundbreaking idea in his thesis that would later be known as the Banach Contraction Principle. This fundamental result established that mapping on a complete metric space has a unique fixed point. As the field evolved, researchers began relaxing the contraction condition, experimenting with other assumptions like continuity, discontinuity in multivalued mappings, cyclic-type contractions, and even the greatest lower bound property to ensure fixed-point existence and uniqueness.

Our thesis is built upon the rich structure of metric spaces, delving into specialized frameworks such as soft metrics, complex-valued metrics, rough metrics, and generalizations of Ciric and Caristi-type theorems. To structure our exploration, we've divided the introduction into thematic segments that reflect key developments in the field.A significant turn in fixed point theory came with the introduction of fuzziness into metric spaces. It began with L. A. Zadeh's influential fuzzy set theory in 1965, which introduced functions mapping from a set (X) to the interval ([0, 1]), representing degrees of membership rather than crisp inclusion. Building on this, Heilpern introduced fuzzy mappings and established fixed-point results for them. In 1975, Kramosil and Michalek further enriched the field with the concept of fuzzy distances—blending fuzzy sets with t-norms to describe relationships between elements in a more nuanced way.

Our research also delves into rough metric spaces and variational inequalities, focusing especially on compact rough metric spaces. These models are crucial for addressing complex phenomena using implicit iterative strategies, and offer powerful tools for both theoretical advancement and practical problem-solving.

Another core component of our investigation is soft S-metric spaces, where we explore fixed-point theorems for soft self-maps. Despite existing literature acknowledging their potential, there remains much to uncover—particularly regarding principles that ensure both existence and uniqueness of solutions.

Our work touches upon several other intricate structures, including soft fuzzy metric spaces, complex-valued metric spaces with modular functions (MF), partially ordered metric spaces, and symmetrical, sequentially dense soft sets. Each of these presents its own blend of challenges and opportunities, and our aim is to provide novel insights and strategies that pave the way for further exploration and interdisciplinary application.

In essence, our research emphasizes the need for refined mathematical frameworks to tackle modern complexities. Fixed-point theory—woven from strands of analysis, topology, and geometry—continues to be a dynamic and inspiring field, and our work contributes to its ongoing evolution.

The concept of Rough Sets, first introduced by Pawlak [195], has been instrumental in addressing the granularity inherent in data. Building on this foundation, researchers like Ramakant Bhardwaj [181] extended the theory into metric spaces, leading to the development of fixed-point theorems in Compact Rough Metric Spaces—a testament to the evolving versatility of these mathematical ideas.

Our work also explores the fascinating field of fractional calculus, which provides a non-local framework for differentiation and integration. This area is particularly powerful in modeling complex processes in science and engineering. We've focused on the application of fixed-point theory to prove the existence and uniqueness of solutions for fuzzy fractional Volterra-Fredholm integro-differential equations—a beautiful intersection of abstract theory and real-world relevance.

Another pillar of our research is variational inequality theory, a mathematical tool introduced in the 1960s that has since found its way into diverse fields like economics, engineering, and finance. The development of general variational inequalities, such as those proposed by Noor , has opened new avenues for solving problems involving multivalued mappings and differential equations—both ordinary and partial. Techniques like the Adams-Bashforth method for Atangana-Baleanu fractional integrals also contribute to these advancements.

Initially, much of this research was grounded in strict contraction mapping conditions. However, recent developments have moved beyond those limitations, exploring alternatives such as cyclic-type contractions, discontinuous mappings, and properties like the greatest lower bound, all crucial in establishing uniqueness of fixed points.

Ultimately, this thesis navigates a rich landscape of metric spaces, including Soft metric, Complex-valued metric, and Rough metric spaces. It also extends classical theorems—such as those by Ćirić and Caristi—into new, generalized forms, pushing the boundaries of what's possible in nonlinear analysis.

Theorem 1.1 (Theorem (1). *"Define a mapping, $\zeta:\rho\to\rho$ and consider $\exists\ 0\leq\omega<1$, and (ρ,ρ) be a complete metric space, such that*

$$\rho(\zeta y,\zeta x)\leq\omega\ \chi(y,x)\ .....\ (3)$$

$$\Rightarrow\rho(\zeta y,\zeta x)\leq\omega\ \max\{\rho(y,\zeta y),\rho(y,x),\rho(x,\zeta x),\rho(y,\zeta x),\rho(x,\zeta y)\}\text{ for all } y,x\in\rho\ ....\ (4)$$

Then inside ρ, ζ admits a unique fixed point.

Definition 1.2 (Definition (1)). *Consider (ρ, ρ, <) be partially ordered complete metric space and distance ρ for any non-increasing sequence* $(y_n)n\in\mathrm{N}$ *in a ρ such that* $\lim_{n\to+\infty}(y_n)$ *exists. Then ∃* $\inf_n y_n$ *and which is equal to* $\lim_{n\to+\infty}(y_n)$. *Then we say that ρ Satisfies above definition (1).*

Theorem 1.3 . *Consider (ρ,ρ,<) be partially ordered complete metric space and distance ρ satisfies definition (1). Lets define a mapping ζ:ρ→ρ be a monotone non- decreasing, consider we have a lower semicontinuous function, η: ρ→ [0, +∞) such that,*

$$\rho(y, \zeta y) \leq \eta(y)-\eta(\zeta y), \text{ whenever } \zeta y<y \text{ for all } y\in\rho \ ......(5)$$

which gives ζ admits a fixed point if and only if ∃ $y_0 \in \rho$ *such that* $\zeta y_0<y_0$.

Definition 1.4. *Topological space with Partially order- Suppose of* (Ξ, *τ*) *is a topological space having a partial order ≼ then for all μ ∈ Ξ, we represented this ≼-principal ideal and principal filter of μ inside* (Ξ, ≼), *respectively, given as*

$$\mu\downarrow = \{u \in \Xi: u \preccurlyeq \mu\} \text{ and } \mu^{\uparrow} = \{u \in \Xi: \mu \preccurlyeq u\}.$$

Consider Ψ *is a nonempty subset of* Ξ *then we have*

$$\Psi\downarrow = \{u \in \Xi: u \preccurlyeq \mu, \text{ for some } \mu \in \Psi\},$$

along with

$$\Psi\uparrow = \{u \in \Xi: \mu \preccurlyeq u, \text{ for some } \mu \in \Psi\}.$$

with any $\Psi \subseteq \Xi$*, we know* $\Psi \subseteq \Psi^{\downarrow} \cap \Psi^{\uparrow}$ *whenever* $\Psi^{\downarrow} = \Psi$*, gives* Ψ *is said to be* $\preccurlyeq$*-decreasing and* $\Psi\uparrow = \Psi$*, gives* Ψ *called as* $\preccurlyeq$*-increasing function.*

In this work, we follow the definition saying that $(\Xi, \tau, \preccurlyeq)$ is a partially ordered topological space, if the graph of $\preccurlyeq$ be a closed subset of $\Xi \times \Xi$ having product topology.

Proposition 1.1.1. *Let's consider* (Ξ, τ) *be a topological space having a partial order* $\preccurlyeq$*. Then we write following condition are equivalent:*

1. $(\Xi, \tau, \preccurlyeq)$ *be a partially ordered topological space (that is* $\preccurlyeq$ *be closed subset inside* $\Xi \times \Xi$*);*
2. *For every* $\mu, y \in \Xi$*, we write* $\mu \preccurlyeq y$ *be a false, which has disjoint neighborhoods V of x and U of y then V is* $\preccurlyeq$ *be a increasing and U is* $\preccurlyeq$ *be a decreasing; Subsequently, (2) gives the following (3) but (3) does not gives (2).*
3. *For all* $\mu \in \Xi$*, both* $\mu^{\downarrow}$ *and* $\mu^{\uparrow}$ *are* τ*-closed.*

Definition 1.1.1. *Topologies on the hyperspaces with partially ordered topological spaces. Consider the* $(\Xi, \tau, \preccurlyeq)$ *is a partially ordered topological space. Suppose* $C(\Xi)$ *represent the collection of every* τ *closed subsets from* Ξ *and* $C_0(\Xi) = C(\Xi)\backslash \{0/\}$*. Denote the set of all* $\preccurlyeq$*-principal ideals in* Ξ *by* $C^{\downarrow}(X) = \{\mu^{\downarrow} : \mu \in \Xi\}$*. By Part (3) in above proposition, we write* $C^{\downarrow}(\Xi) \subseteq C(\Xi)$*.*

Remark 1.5. *Topology* τ_F *on* $C(\Xi)$*, we write every* τ*-open subset O of* Ξ *and all* τ*-compact subset D of* Ξ*, we show the following subsets from* $C(\Xi)$:

$$O^{-} = \{\Psi \in C(\Xi) : \Psi \cap O \neq 0/\} \text{ and } (\Xi \backslash D)^{+} = \{\Psi \in C(\Xi) : \Psi \cap D = 0/\}.$$

Then the set of all subsets O^{-} from O running through collection from all τ-open subsets and $(\Xi\backslash D)^{+}$for D through all τ-compact subsets of Ξ the base of a topology C(Ξ), which said to the Fell topology on C(Ξ) and is written as τ_F.

Definition 1.1.2. *Let the (Ξ,τ) be Hausdorff topological space and the Fell topological hyperspace $(C(\Xi),\tau_F)$ having many useful properties. We list two of them as give below (P1) $(C(\Xi),\tau_F)$ is compact; (P2) $\mu \rightarrow \{\mu\}$ embeds X in C(Ξ).*

Definition 1.1.3. *Topology τ_V on C(Ξ). The Vietoris topology τ_V on C(Ξ) is induced by the following base $\{O^{-} : O$ is an τ-open subset of $\Xi\}\cup\{(\Xi\backslash E)^{+} : E$ is a τ-closed subset of $\Xi\}$. Then $(C(\Xi),\tau_V)$ is called the Vietoris topological hyperspace of (Ξ,τ).*

Definition 1.1.4. *Suppose the Hausdorff topology τ_H on C(Ξ). In particular, (Ξ,τ) is a metric space and the topology τ is induced by a metric d on Ξ. The Hausdorff metric H on C(Ξ) is defined, for any distinct A, B ∈C(Ξ) as*

$$H(\Psi,\tau) = \max\left\{\sup_{a\in\Psi}\left(\inf_{b\in\tau} d(a,b)\right), \sup_{b\in\tau}\left(\inf_{a\in\Psi} d(b,a)\right)\right\}.$$

Then we say here Hausdorff metric H induces the Hausdorff topology on C(Ξ).

. Fixed-point theory has found a rich playground in specialized metric spaces, and fuzzy set theory has proven to be a powerful ally in handling uncertainty and ambiguity in data. The idea of fuzzy metric spaces, first shaped by Kramosil and Michalek and later developed by George and Veeramani, marked a turning point in how we approach imprecise information. Molodtsov further advanced the field by introducing soft sets—a framework designed specifically for parameter-driven uncertainty.

A cornerstone of this exploration is the Banach Contraction Principle, a pivotal result from functional analysis. This principle has inspired new ways of thinking, including efforts to develop a contraction mapping theory tailored to soft fuzzy metric spaces—a kind of "soft" generalization of traditional fuzzy metric concepts.

The theory expanded to include cyclic contractions, thanks to work by George and others, who showed how these ideas could ensure the existence and uniqueness of fixed points for more complex operators. An exciting innovation emerged from soft set theory—the concept of soft single points—which opened new possibilities in soft topology. These soft real points aren't just abstract constructs; they lay the foundation for new algebraic and topological structures, and they help broaden the reach of classic fixed-point theorems.

While Cantor laid the foundation of set theory back in 1874, today's challenges in economics, social sciences, and medicine often come with layers of uncertainty. Fuzzy set theory (pioneered by Zadeh) and soft set theory (by Molodtsov) offer distinct mathematical lenses to confront these challenges, going where traditional probability theory falls short.

This mathematical machinery doesn't just live in theory—it's at work in internet topology, cyberspace models, vehicle routing logistics, communication engineering, genetics, control theory, and more. It's how we translate abstract fixed-point results into real-world problem-solving.

Another fascinating area is the study of bicomplex numbers—a generalization of complex numbers that supports a full suite of mathematical operations. Thanks to contributions from researchers like Luna-Elizarrarás and others, bicomplex algebra has begun influencing everything from mathematical theory to applied science and technology.

Ongoing research in all these areas promises not only deeper understanding but also exciting breakthroughs in a wide range of scientific and practical fields.

Advancements in Mathematical Analysis: A Journey from Fractional Calculus to Variational Inequality

Mathematical analysis has evolved dramatically over the centuries, with new theories and approaches constantly reshaping how we understand complex systems. Among the most fascinating developments are those in Fractional Calculus, Integral Equations, Metric Fixed-Point Theory, Rough Set Theory, and Variational Inequality Theory—each contributing its own unique perspective to science and engineering.

- Fractional Calculus (FC)

Imagine stretching the boundaries of traditional calculus to explore derivatives and integrals of non-integer order. That's precisely what fractional calculus does—a concept that traces back to brilliant minds like Leibniz, Euler, and Riemann. Unlike classical differential equations, fractional differential equations (FDEs) allow us to model systems with memory or hereditary properties, making them ideal for describing real-world dynamics. Researchers have built on this foundation in exciting ways, like applying fixed point theory to fuzzy-valued fractional equations, opening doors to modeling uncertainty in fields from physics to finance.

- Integral Equations (IE)

Integral equations are the unsung heroes behind many scientific breakthroughs. Whether it's heat transfer, electromagnetism, or quantum mechanics, these equations help describe how systems evolve over space and time. The Urysohn integral equation, in particular, continues to be a key area of research, touching disciplines as diverse as biology and engineering. Mathematicians like Noor have connected these equations with variational inequalities, offering fresh insights into how we model complex interactions.

- Metric Fixed Point Theory (MFT)

At the heart of this theory lies the Banach contraction principle—a powerful idea that guarantees the existence of fixed points under specific conditions. Over time, mathematicians have expanded on this with principles from Rakotch, Kannan, Boyd and Wong, and others, giving us a rich toolkit for analyzing nonlinear systems. These theoretical developments don't just live on chalkboards—they help model everything from biological systems to game theory and computer algorithms.

- Rough Set Theory (RST)

Developed by Zdzisław Pawlak in the 1980s, rough set theory gives us the mathematical machinery to deal with imprecise or incomplete information—a common challenge in today's data-driven world. Its applications have grown rapidly, from rough vector spaces and rings to novel adaptations in information sciences. Ongoing efforts to generalize these structures showcase just how versatile rough sets can be in handling real-world complexity.

- Variational Inequality Theory (VIT)

This is where mathematics meets optimization. Initially grounded in pure math, variational inequality theory now spans numerous applied domains, from engineering and finance to social sciences. The field gained momentum with Noor's concept of general variational inequalities (GVI), which offered a broader framework for solving problems involving equilibrium and optimization. These tools have become central to tackling large-scale, real-world issues.

The progress in mathematical analysis, from fractional calculus to variational inequality theory, reflects a dynamic and interdisciplinary landscape. These theories and methods continue to be at the forefront of addressing complex problems in various scientific disciplines, showcasing the enduring impact of mathematical advancements on our understanding of the natural world.

Chapter 2

Review of Literature

2.1 Introduction

The fractional calculus and calculus of integral equations have been foundational topics in mathematics for over a century. While a decade might seem like a substantial period, these areas have continually evolved, acquiring new structures and finding effective applications in various branches of mathematics, including fixed point and fuzzy theory. In 1999, the concept of soft sets was introduced by the Russian scientist Molodtsov, providing a versatile mathematical tool for capturing uncertainties. Since then, soft set theory has gained appreciable attention due to its broad range of potential applications. Maji et al. implemented soft sets in decision-making problems, opening the door for further exploration. Subsequently, Babitha and Sunil introduced the notions of soft set relation and function, delving into related concepts. Building upon their work, Maji et al. witnessed transformative developments from scholars like Sezgin and Atagun, who made significant contributions to the field.

Atangana-Baleanu in, introduced a new type of fractional derivative focusing on non-singular/ local kernels. Building on this, S. Shinde presented a complex-valued version, addressing the existence and common solutions for second-order nonlinear boundary value problems. Further applications of fixed-point results for multivalued mappings in the context of complex-valued metric spaces ($CVMS$) were explored. In 2019, H. Kumar delved into a class of two-variable sequences of functions satisfying Abel's integral equation, with subsequent work in 2017, where H. Kumar et. al. studied a fractional nonlinear biological model problem and its approximate solutions through Volterra integral equations. Various generalizations of the Atangana-Baleanu fractional derivative, such as AB derivative , AB derivative via MHD channel flow, and ABRL type. The Atangana-Baleanu fractional integral was also revisited, and the fractional Adams-Bashforth method was applied to integral type equations, demonstrating promising uniqueness of solutions in different fields. In the study of fixed point problem-solving methodologies, Banach provided the fixed-point principle, challenging the limitations of classical set theory when addressing real-world challenges embedded in uncertainty. This led researchers to explore alternative theories like probability, fuzzy sets, rough sets, and soft sets, we can see (2.1) uniqueness solution as per contraction condition varies as per the varying value of variable n The research further builds upon comprehensive explorations of rough sets, fuzzy metric spaces

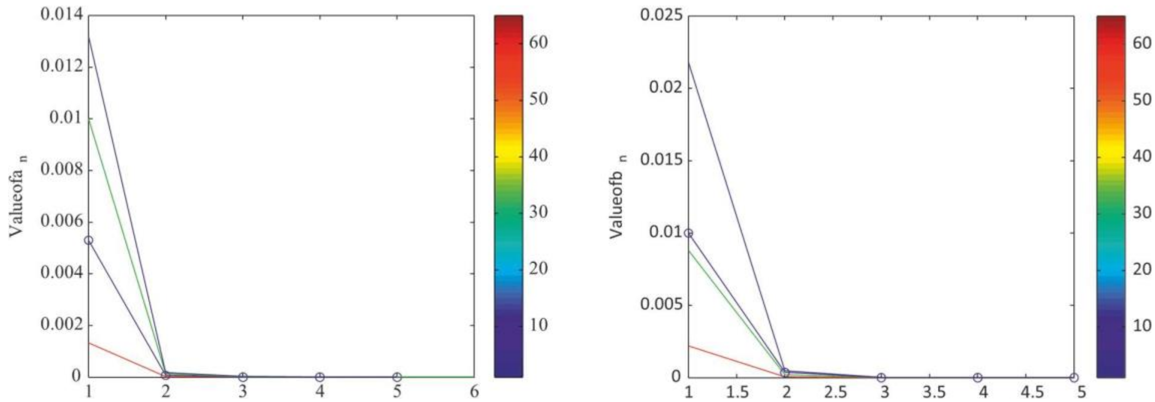

Figure 2.1: Uniqueness solution iteration for Integral equation
Fixed point iteration

Developments in rough sets initiated by Z. A. Pawlak in 1982 catalysed a prolific body of research into various mathematical structures.

Variational inequality theory, developed since the 1960s, has found applications across diverse fields, with numerous numerical approaches and generalizations enriching its adaptability in problem solving contexts. In 1988, Noor introduced a diverse class of variational inequalities (GVI), contributing innovative and integrated methods applicable across scientific fields. The research explores implicit iterative approaches within the framework of variational inequalities, introducing convergence criteria and offering valuable insights for mathematical research. The convergence analysis and a numerical example illustrate the practical implementation of these novel findings. Das and Samanta proposed the concept of soft real sets and soft real numbers in 2012, exploring their fundamental characteristics. They also introduced the concept of soft metric space, opening new avenues for investigation. At the core of metric fixed-point theory lies the influential Banach contraction theorem, which has been extensively studied by numerous scholars. Over time, these studies have led to modifications, extensions, and practical implementations of the Banach fixed-point principle. Wardowski's work in 2013 introduced the notion of a soft mapping and its fixed coordinates. Further developments by Abbas et al. presented the first fixed-point theory of soft metric spaces in 2016. Soft metric spaces exhibit various generalizations that contribute to the understanding of fixed-point theorems. Notable among these is the proof of the fixed-point theorem in the soft G metric space by Guler et al. In 2016, defining the soft cone metric space and exploring its implications. Wadkar et al. introduced the concept of soft b-metric in 2017, along with accompanying fixed-point results within the context of soft b-metric fields. Further works on this topic can be found in the literature. Rakotch pioneered a distinct contraction principle, incorporating a function in lieu of a Lipschitz constant as seen in the Banach contraction principle. Boyd and Wong extended the Rakotch contraction principle, and Kannan introduced a contraction principle characterizing metric completeness, inspiring numerous mathematicians to propose diverse contraction principles. Among the eminent classical contraction theorems are the Meir and Keeler contraction principle, Chatterjea contraction principle, Reich contraction principle, Hardy and Rogers contraction principle, Ciric contraction principle, and Caristi contraction principle. In 2012, Wardowski elevated the Banach contraction principle by employing an auxiliary nonlinear function, marking a significant milestone in metric fixed-point theory. The F-contraction principle has since undergone thorough exploration and generalization across various abstract spaces, evident in works and related references. In 2012, Das and Samanta proposed the concept of soft real sets and soft real numbers, exploring their fundamental characteristics. They also introduced the notion of a soft metric space, opening up new avenues for investigation. At the core of metric fixed-point theory lies the influential Banach contraction theorem, extensively studied by numerous scholars. Over time, these studies have led to modifications, extensions, and practical implementations of the Banach fixed-point principle.

Wardowski's work in 2012 introduced the notion of a soft mapping and its fixed coordinates. Further developments by Abbas et al. presented the first fixed-point theory of soft metric spaces in 2016. Soft metric spaces exhibit various generalizations contributing to the understanding of fixed-point theorems. Notably, Guler et al. proved the fixed-point theorem in the soft G metric space in 2016, defining the soft cone metric space and exploring its implications. In 2017, Wadkar et al. introduced the concept of a soft b-metric, along with accompanying fixed-point results within the context of soft b-metric fields. Additional works on this topic can be found in the literature. The metric fixed-point principle is integral to advancing metric fixed-point theory, enhancing theoretical understanding, and ensuring the existence of solutions to mathematical models. Rakotch pioneered a distinct contraction principle, incorporating a function in lieu of a Lipschitz constant, as seen in the Banach contraction principle. Boyd and Wong extended the Rakotch contraction principle, and Kannan introduced a contraction principle characterizing metric completeness, inspiring numerous mathematicians to propose diverse contraction principles. Among the eminent classical contraction theorems are the Meir and Keeler contraction principle, Chatterjea contraction principle, Reich contraction principle, Hardy and Rogers contraction principle , Ciric contraction principle, and Caristi contraction principle. In 2012, Wardowski elevated the Banach contraction principle by employing an auxiliary nonlinear function, marking a significant milestone in metric fixed-point theory. The F-contraction principle has since undergone thorough exploration and generalization across various abstract spaces, as evident in works and related references. In the recent era, the combination of soft set theory and fixed point theories, such as Ciric and Caristi fixed points , has emerged as a fascinating branch of study. The foundational concepts of soft set theory, introduced by Molodtsov in 1999, and fixed point principles by Banach in 1922 , have been extended and generalized, gaining significant attention. Notably, Ciric and Caristi fixed point theories have been extensively explored in the literature. Das and Samanta contributed to this field by defining real soft numbers and real soft sets, along with their properties. They further delved into the concept of soft metric space. Abbas et al. introduced the notion of soft contraction mapping, which relies on soft metric and soft elements. Wadkar et al. shared their thoughts on fixed point theory in soft metric space, and subsequent research by Wadkar, Bhardwaj, and Sharaff generalized the fixed-point theorems in soft metric spaces.

The application of soft set theory has seen significant developments, with Maji et al. extending soft theory results to decision-making problems. T. M. Al-shami introduced somewhat open sets in 2022, providing soft separation axioms along with medical applications to nutrition. Additionally, Das and Samanta contributed to the notion of soft metric space based on the soft point of given soft sets. Researchers like S. Rathee, B. Sadi, and R. I. Sabri explored various aspects of soft metric spaces, including contraction fixed point results. In the context of *SFZMS,* a soft fuzzy metric space, the study addresses the absence of a generalization of the Banach contraction mapping principle. To fill this research gap, the authors establish the ϖ-contraction mapping with a ψ-function and demonstrate that their contraction map holds a unique soft fuzzy fixed point. The research study provides proofs for the existence and uniqueness of a soft fixed point in *SFZMS* , supported by validating examples. This work is structured with preliminary sections, an introduction to contraction mapping principles, the main result establishment, validation through examples, and a concluding section. Apart from the generalization of contractive conditions, researchers have invested ample time in solving diverse equations using the fixed point method. Abdel jawad et al. extended the study of double controlled metric spaces to complex double controlled metric spaces, presenting a contraction theorem. Furthermore, the fixed point method has been applied to solve various problems, including linear systems of equations, differential equations, fractional differential and integral equations, complex Riemann–Liouville fractional operators, Atangana–Baleanu fractional integral operators, and nonlinear telegraph equations. Fixed point theory has established itself as a fundamental and versatile tool with applications across various mathematical disciplines, including engineering and computer sciences. This theory acts as a bridge connecting different areas such as topology, geometry, and algebra. Banach laid the foundation for fixed point theory by introducing the concept of contraction mapping in a complete metric space . Since then, researchers have employed diverse techniques to study and extend Banach's contraction theory, as evidenced by a significant body of work. Fixed point approaches play a crucial role in validating solutions to different types of equations, including integral equations, differential equations, and fractional differential equations. The concept of metric space has undergone generalization over time. Harandi introduced metric-like spaces, Azam et al. extended the notion to complex-valued metric spaces, and subsequent researchers, such as Hosseni, Karizaki, Bakhtin, and Czerwik introduced variations like b-metric and complex-valued controlled metric type space. Abdel jawad et al. contributed the notion of double controlled metric type space (DCMLS), building upon the work of Kamran, Mlaiki, and others.

Uncertainty is inherent in practical problems across diverse fields like economics, social science, and health. Conventional mathematical techniques often fall short in handling uncertainties, leading to the exploration of alternative theoretical frameworks, such as soft sets theory. Molodtsov developed the theory of soft sets, providing a novel approach to managing uncertainties in various disciplines. The theory of soft sets has found applications in economics, engineering, social science, and medicine, with breakthroughs and innovations continuing to emerge, in 2022, He introduced the notion of the Split equilibrium problem (*SEQP*), where the solution to an equilibrium problem (*EP*) has an image that is also the solution to another *EP* under a given bounded linear operator. This involves considering two different real Hilbert spaces ($\Im^1 and\ \Im^2$) and closed convex subsets ($C\ and\ Q$). This study delves into the realm of fixed-point theory, exploring its wide-ranging applications in geology, biology, mechanics, economics, and the modeling of nonlinear integral and differential equations. The study of fixed points has become a powerful tool in non-linear analysis, contributing significantly to fields like numerical analysis and differential equations. Geraghty's real-valued mapping is employed, and Das and Gupta have extended the Banach contraction principle for rational-type contractive conditions under fixed point metric spaces. This work also discusses the generalization of complex-valued fixed-point theory, exploring applications in Urysohn integral equations under complex-valued metric spaces.

In recent times, Ciric and Caristi-type theorems for fixed points have gained considerable attention due to their wide applications. The study involves the application of various generalizations of Banach's principle in metric spaces with partial order. Caristi's known results in the literature form the basis for several theorems, stating conditions for mappings in complete spaces with lower semi-continuous maps. The Banach contraction mapping theorem remains a prominent tool for solving problems in nonlinear analysis, calculus, and fuzzy theory. Generalizations of this theorem in different metric spaces have been pursued by various authors, contributing to complex-valued metric spaces, certain rational expressions and generalized contractive conditions. This study further explores the application of these results to the solution of multivalued mappings and second-order nonlinear boundary value problems in the context of complex-valued metric spaces. Theoretical results are then applied to prove common fixed-point solutions in the study of multivalued mappings and second-order nonlinear boundary value problems.

In this dissertation, we've looked into whether solutions to integral equations exist and if those solutions are unique, using fixed point theory as our main tool. By applying different fixed point theorems—especially Banach's contraction principle and related ideas—we've figured out clear conditions that guarantee solutions for certain types of integral equations exist and are one-of-a-kind.

The step-by-step methods we developed to approximate these solutions not only show they get closer to the real answer but also stay stable under conditions where things shrink in a predictable way. These fixed point methods give us a hands-on way to solve integral equations, which is really handy in real-world situations where finding exact solutions mathematically might be tough or even impossible.

Furthermore, our results on how these methods converge prove that the fixed point approach is a strong and adaptable tool in studying non-linear problems. This method works for a wide range of equations, like Volterra and Fredholm types, and can be tweaked to handle extra complications such as kernels that aren't smooth or operators in more abstract settings like metric or Banach spaces.

To wrap up, combining fixed point theory with the study of integral equations provides both a solid theoretical base and a practical, step-by-step way to tackle non-linear problems in areas like mathematical analysis, physics, and engineering.

Chapter 3

A Topological fixed point uniqueness approach to Non-linear second order Boundary value problem and multi-valued mappings in Complex valued metric

3.1 Introduction

Banach contraction mapping theorem is a prominent tool for solving problem in nonlinear analysis, Calculus, Fuzzy Theory and so on. This principle used to establish existence and uniqueness of common solution for nonlinear integral equation and several other fields, many authors generalized this theorem in different metric spaces within that In 2011 Azam, Khan and Fisher present the notion of complex valued metric space and given some result for pair of mapping which is contraction condition satisfying a rational expression. After this establishment Rouzkard and Imdad generalized some common fixed point theorems involving certain rational expressions. In 2013 Ahmad, Klin eam and Azam studied multivalued mapping under generalized contractive condition subsequently Azam, Al Rawashden proved a common fixed point for multivalued mapping. Ahmad, Klin-eam, Azam & Azam, Ahmad, Kumam defined generalized Housdorff metric function in the setting of CVMS and established common fixed point result for multivalued mapping.

Das and gupta generalized banach contraction principle for rational type contractive inequality in fixed point metric space. Afterwards several researchers generalized and extended the aforesaid work with the help of different rational contraction under self and multivalued mapping in the context of CVMS, in addition Sinthunavarat et.al. studied a common solution to the Urysohn integral equation under CVMS. In this study we studied results from and so on, afterward we generalized Result from literature given by Azam, Jamshaid Ahmad, Klin-Eam as following, Theorem 3.1. *Consider the complete complex valued metric space* (Θ, ð) *and the function* ζ, ξ:

$\Theta \rightarrow CB(\Theta)$ *be multivalued mappings having global property such that*

$$[\eta_1 . ð(\mu, v) + \frac{\eta 2 . ð(\text{ఔ}, £\text{ఔ}) . ð(\mu, £\mu) + \eta 3 ð(\text{ఔ}, £\mu) . ð(\mu, £\text{ఔ})]}{1 + ð(\mu, \text{ఔ})} \in_{\omega} (\zeta\mu, \xi v)$$

for every μ, $v \in \Theta$ *and* $\eta 1$, $\eta 2$, $\eta 3$ *are non-negative real number with* $\eta_1 + \eta_2 + \eta_3 < 1$
Then ζ, ξ *have a common fixed point.*

With inspiring all above results form literature, we prove common fixed-point solution to the multivalued mapping and second order nonlinear boundary value problem. In our first section we go through some important results and definition from literature.

3.2 Findings

The main aim of this chapter is to work on the application part of CVMS. In this work we have demonstrated some common fixed results and then we deal primarily with two parts of applications, part(I) Complex valued version of existence and common solution for second order nonlinear boundary value problem using greens function,

part (II) Application of fixed-point results for multivalued mapping in setting of CVMS without using notion of continuity. Eventually several equivalent results and examples are presented to sustain our Main result.

3.3 Preliminaries

$$\mu''(x) = Im\,(x, \mu(x), \mu'(x)), when\ x \in [0, Ț], Ț > 0$$
$$\mu(x1) = \mu 1,$$
$$\mu(x2) = \mu 2, when\ x1, x2 \in [0, Ț].$$

Definition 3.2. *Suppose a partial order* $\precsim$ *defined on a complex number*(C) *as,* $\mu \precsim v$ *iff Real part of (μ) ≤ Real part of (v) ; Imaginary part of (μ) ≤ Imaginary part of (v). It follows,* $\mu \leq v$

- *Real part of (μ)<Real part of (v) ; Imaginary part of (μ) < Imaginary part of (v).*
- *Real part of (μ) = Real part of (v) ; Imaginary part of (μ) = Imaginary part of (v).*
- *Real part of (μ) < Real part of (v) ; Imaginary part of (μ) = Imaginary part of (v).*
- *Real part of (μ) = Real part of (v) ; Imaginary part of (μ) < Imaginary part of (v).*

Definition 3.3. *Let* Θ *be non empty set* & *assume that the self-mapping* ð *:* Θ→Θ *said to be complex valued metric if* ð *satisfy following condition,*

1. $0 \precsim ð(\mu, v)$ *every* $\mu, v \in \Theta$ *and* $ð(\mu, v) = 0$ *if and only if* $\mu = v$
2. $ð(\mu, v) = ð(v, \mu)$ *every* $\mu, v \in \Theta$
3. $ð(\mu, v) \precsim ð(\mu, \rho) + ð(\rho, v)$ *for every* $\mu, v, \rho \in \Theta$*, then* (Θ,ð) *is called complex valued metric space.*

Example 3.4. *Suppose* Θ = C *and the mapping* ð : Θ×Θ → C *which has* $ð(\mu, v) = e^{i\alpha} | \mu - v |$ *Where,* $\alpha \in [0, \frac{\pi}{2}]$*, Then (*Θ, ð) *be complex vlaued metric space.*

Example 3.5. *Assume* Θ = C *and the mapping* ð : C×C → C *defined as following* $ð(\mu, v) =| \kappa_1 - \kappa_2 | + \imath | \imath_1 - \imath_2 |$ *Where,* $\mu = \kappa_1 + \imath\imath_1$ $v = \kappa_2 + \imath\imath_2$*. Then (*Θ, ð) *be complex vlaued metric space.*

Definition 3.6. *Assume* $\{\mu_r\}$ *be a sequence in a complex valued metric space* (Θ,ð) *and* $\mu \in \Theta$*, Then μ is a limit point of* $\{\mu_r\}$ *if every* $\varepsilon \in$ C *there exist* $r_0 \in$ N *such that* $ð(\{\mu_r\}, r) \prec \varepsilon, \forall r \succ r_0$ *that is* $\lim r \to \infty, \mu_r = r$.

Definition 3.7. *[3] Assume* $\{\mu_r\}$ *be a sequence in a complex valued metric space* (Θ,ð) *and* $\mu \in \Theta$*,* *Then* $\{\mu_r\}$ *is a cauchy sequence if for any* $\varepsilon \in$ C *there exist* $r_0 \in$ N *such that* $ð(\mu_r,\mu_{r+s}) \prec \varepsilon, \forall r \succ r_0$ *and* $s \in$ N .

Definition 3.8. *Assume* $\{\mu_r\}$ *be a sequence in a complex valued metric space* (Θ,ð) *and* $\mu \in \Theta$*, Then* (Θ,ð) *is said to be a complete complex valued metric space if every Cauchy sequence is convergent in* (Θ,ð).

Proposition 3.3.1. *Suppose* (Θ,ð) *be a complex valued metric space. Then the sequence* $\{\mu_r\}$ *in* Θ *Converges to* μ *if and only if* $|ð(\mu_r,\mu)| \to 0$ *as* $r \to \infty$.

Proposition 3.3.2. *Suppose* (Θ,ð) *be a complex valued metric space. Then the sequence* $\{\mu_r\}$ *in* Θ *is a cauchy sequence if and only if* $|ð(\mu_r,\mu_{r+s})| \to 0$ *as* $r \to \infty$ *where* $s \in$ N.

Definition 3.9. *Topological space with Partially order- Suppose of* (Ξ,τ) *is a topological space having a partial order* $\preccurlyeq$ *then for all* $\mu \in \Xi$*, we represented this* $\preccurlyeq$*-principal ideal and principal filter of* μ *inside* $(\Xi,\preccurlyeq)$*, respectively, given as*

$$\mu\downarrow = \{u \in \Xi : u \preccurlyeq \mu\} \text{ and } \mu^{\uparrow} = \{u \in \Xi : \mu \preccurlyeq u\}.$$

Consider Ψ *is a nonempty subset of* Ξ *then we have*

$$\Psi\downarrow = \{u \in \Xi : u \preccurlyeq \mu, \text{ for some } \mu \in \Psi\},$$

along with

$$\Psi\uparrow = \{u \in \Xi : \mu \preccurlyeq u, \text{ for some } \mu \in \Psi\}.$$

with any $\Psi \subseteq \Xi$*, we know* $\Psi \subseteq \Psi^{\downarrow} \cap \Psi^{\uparrow}$ *whenever* $\Psi^{\downarrow} = \Psi$*, gives* Ψ *is said to be* $\preccurlyeq$*-decreasing and* $\Psi\uparrow = \Psi$*, gives* Ψ *called as* $\preccurlyeq$*-increasing function.*

In this work, we follow the definition saying that $(\Xi,\tau,\preccurlyeq)$ is a partially ordered topological space, if the graph of $\preccurlyeq$ be a closed subset of $\Xi\times\Xi$ having product topology.

Proposition 3.3.3. *Let's consider* (Ξ,τ) *be a topological space having a partial order* $\preccurlyeq$*. Then we write following condition are equivalent:*

1. $(\Xi,\tau,\preccurlyeq)$ *be a partially ordered topological space (that is* $\preccurlyeq$ *be closed subset inside* $\Xi\times\Xi$*);*
2. *For every* $\mu,y \in \Xi$*, we write* $\mu \preccurlyeq y$ *be a false, which has disjoint neighborhoods* V *of* x *and* U *of* y *then* V *is* $\preccurlyeq$ *be a increasing and* U *is* $\preccurlyeq$ *be a decreasing; Subsequently, (2) gives the following (3) but (3) does not gives (2).*
3. *For all* $\mu \in \Xi$*, both* $\mu^{\downarrow}$ *and* $\mu^{\uparrow}$ *are* τ*-closed.*

Definition 3.3.1. *Topologies on the hyperspaces with partially ordered topological spaces. Consider the* $(\Xi,\tau,\preccurlyeq)$ *is a partially ordered topological space. Suppose* $C(\Xi)$ *represent the collection of every τclosed subsets from* Ξ *and* $C_0(\Xi) = C(\Xi)\setminus\{0/\}$*. Denote the set of all* $\preccurlyeq$*-principal ideals in* Ξ *by* $C^{\downarrow}(X) = \{\mu^{\downarrow} : \mu \in \Xi\}$*. By Part (3) in above proposition, we write* $C^{\downarrow}(\Xi) \subseteq C(\Xi)$*.*

Remark 3.10. *Topology* τ_F *on* $C(\Xi)$*, we write every* τ*-open subset* O *of* Ξ *and all* τ*-compact subset* D *of* Ξ*, we show the following subsets from* $C(\Xi)$ *:*

$$O^{-} = \{\Psi \in C(\Xi) : \Psi\cap O \neq 0/\} \text{ and } (\Xi\setminus D)^{+} = \{\Psi \in C(\Xi) : \Psi\cap D = 0/\}.$$

Then the set of all subsets O^{-} *from* O *running through collection from all* τ*-open subsets and* $(\Xi\setminus D)^{+}$*for* D *through all* τ*-compact subsets of* Ξ *the base of a topology* $C(\Xi)$*, which said to the Fell topology on* $C(\Xi)$ *and is written as* τ_F*.*

Definition 3.3.2. *Let the* (Ξ,τ) *be Hausdorff topological space and the Fell topological hyperspace* $(C(\Xi),\tau_F)$ *having many useful properties. We list two of them as give below (P1)* $(C(\Xi),\tau_F)$ *is compact; (P2)* $\mu \to \{\mu\}$ *embeds* X *in* $C(\Xi)$*.*

Definition 3.3.3. *Topology* τ_V *on* $C(\Xi)$. *The Vietoris topology* τ_V *on* $C(\Xi)$ *is induced by the following base* $\{O^- : O$ *is an* τ*-open subset of* $\Xi\} \cup \{(\Xi \backslash E)^+ : E$ *is a* τ*-closed subset of* $\Xi\}$. *Then* $(C(\Xi), \tau_V)$ *is called the Vietoris topological hyperspace of* (Ξ, τ).

Definition 3.3.4. *Suppose the Hausdorff topology* τ_H *on* $C(\Xi)$. *In particular,* (Ξ, τ) *is a metric space and the topology* τ *is induced by a metric* d *on* Ξ. *The Hausdorff metric* H *on* $C(\Xi)$ *is defined, for any distinct* $A, B \in C(\Xi)$ *as*

$$H(\Psi, \tau) = \max\left\{\sup_{a\in\Psi}\left(\inf_{b\in\tau} d(a,b)\right), \sup_{b\in\tau}\left(\inf_{a\in\Psi} d(b,a)\right)\right\}.$$

Then we say here Hausdorff metric H *induces the Hausdorff topology on* $C(\Xi)$.

Chapter 4

Results of fixed point in partially ordered soft metric space using topological space

4.1 Introduction

In a recent era soft set along with a fixed-point theory like ciric and caristi type fixed point become one of the fascinating branche of study. The notion of three novel concepts soft set which was asserted by moldostov in 1999, Fixed point principle introduce by Banach in 1992 and one of the extensive concept with generalization of fixed point who got lot more attention which is literature of ciric caristi fixed point which stated as,$(\zeta,\varphi) : (\tilde{\aleph},\tilde{\delta},\rho)\rightarrow(\tilde{\aleph},\tilde{\delta},\rho)$ provided $(\tilde{\aleph},\tilde{\delta},\rho)$ is complete and their exist a lower semi- continuous map, $\eta : (\tilde{\aleph},\tilde{\delta},\rho) \rightarrow [0,+\infty)$ such that,

$$(4.1) \qquad \tilde{\delta}(\tilde{u}_{\omega},(\zeta,\varphi)(\tilde{u}_{\omega})) \leq \eta(\tilde{u}_{\omega})-\eta((\zeta,\varphi)(\tilde{u}_{\omega}))\forall \tilde{u}_{\omega} \in (\tilde{\aleph},\tilde{\delta},\rho).$$

Which gives (ζ,φ)admits a fixed point. Das and Samanta defined real soft number and real soft set along with their properties later on they worked concept of soft metric space. The notion of soft contraction mapping depend on soft metric and soft element which defined by Abbas et al. and worked soft contraction mapping and soft Banach contraction principle with different rule. Wadkar et. al. shared his thought about fixed point theory in soft metric space. Then later this work generalized by Wadkar, Bhardwaj and Sharaff and gives coupled fixed point theorem in soft metric space and then generalized contraction mapping. First we recall some definitions about soft set and from literature we review some important result. Let's consider $\tilde{\aleph}$ -be an initial universe set, E- be the set of˜ parameter and $2^{\tilde{u}}$ be the collection of all subsets of $\tilde{\aleph}$. Subsequently $\tilde{\Re}$ - be a set of Real number, $\tilde{Z}$-be the set of all Integer, $\tilde{N}$-set of Natural number.

Definition 4.1. *Topological space with Partially order- Suppose of* (Ξ,τ) *is a topological space having a partial order* $\preccurlyeq$ *then for all* $\mu \in \Xi$*, we represented this* $\preccurlyeq$*-principal ideal and principal filter of* μ *inside* $(\Xi,\preccurlyeq)$*, respectively, given as*

$$\mu\downarrow = \{u \in \Xi : u \preccurlyeq \mu\} \text{ and } \mu^{\uparrow} = \{u \in \Xi : \mu \preccurlyeq u\}.$$

Consider Ψ *is a nonempty subset of* Ξ *then we have*

$$\Psi\downarrow = \{u \in \Xi : u \preccurlyeq \mu, \text{ for some } \mu \in \Psi\},$$

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with any $\Psi \subseteq \Xi$*, we know* $\Psi \subseteq \Psi^{\downarrow} \cap \Psi^{\uparrow}$ *whenever* $\Psi^{\downarrow} = \Psi$*, gives* Ψ *is said to be* $\preccurlyeq$*-decreasing and* $\Psi\uparrow = \Psi$*, gives* Ψ *called as* $\preccurlyeq$*-increasing function.*

In this work, we follow the definition saying that $(\Xi,\tau,\preccurlyeq)$ is a partially ordered topological space, if the graph of $\preccurlyeq$ be a closed subset of $\Xi\times\Xi$ having product topology.

Proposition 4.1.1. *Let's consider* (Ξ,τ) *be a topological space having a partial order* $\preccurlyeq$*. Then we write following condition are equivalent:*

1. $(\Xi,\tau,\preccurlyeq)$ *be a partially ordered topological space (that is* $\preccurlyeq$ *be closed subset inside* $\Xi\times\Xi$*);*

2. *For every $\mu,y \in \Xi$, we write $\mu \preccurlyeq y$ be a false, which has disjoint neighborhoods V of x and U of y then V is $\preccurlyeq$ be a increasing and U is $\preccurlyeq$ be a decreasing; Subsequently, (2) gives the following (3) but (3) does not gives (2).*
3. *For all $\mu \in \Xi$, both $\mu^{\downarrow}$ and $\mu^{\uparrow}$ are τ-closed.*

Definition 4.1.1. *Topologies on the hyperspaces with partially ordered topological spaces. Consider the $(\Xi,\tau,\preccurlyeq)$ is a partially ordered topological space. Suppose $C(\Xi)$ represent the collection of every τclosed subsets from Ξ and $C_0(\Xi) = C(\Xi)\setminus\{0/\}$. Denote the set of all $\preccurlyeq$-principal ideals in Ξ by $C^{\downarrow}(X) = \{\mu^{\downarrow} : \mu \in \Xi\}$. By Part (3) in above proposition, we write $C^{\downarrow}(\Xi) \subseteq C(\Xi)$.*

Remark 4.2. *Topology τ_F on $C(\Xi)$, we write every τ-open subset O of Ξ and all τ-compact subset D of Ξ, we show the following subsets from $C(\Xi)$:*

$$O^{-} = \{\Psi \in C(\Xi) : \Psi \cap O \neq 0/\} \text{ and } (\Xi \setminus D)^{+} = \{\Psi \in C(\Xi) : \Psi \cap D = 0/\}.$$

Then the set of all subsets O^{-} from O running through collection from all τ-open subsets and $(\Xi\setminus D)^{+}$for D through all τ-compact subsets of Ξ the base of a topology $C(\Xi)$, which said to the Fell topology on $C(\Xi)$ and is written as τ_F.

Definition 4.1.2. *Let the (Ξ,τ) be Hausdorff topological space and the Fell topological hyperspace $(C(\Xi),\tau_F)$ having many useful properties. We list two of them as give below (P1) $(C(\Xi),\tau_F)$ is compact; (P2) $\mu \rightarrow \{\mu\}$ embeds X in $C(\Xi)$.*

Definition 4.1.3. *Topology τ_V on $C(\Xi)$. The Vietoris topology τ_V on $C(\Xi)$ is induced by the following base $\{O^{-} : O$ is an τ-open subset of $\Xi\} \cup \{(\Xi\setminus E)^{+} : E$ is a τ-closed subset of $\Xi\}$. Then $(C(\Xi),\tau_V)$ is called the Vietoris topological hyperspace of (Ξ,τ).*

Definition 4.1.4. *Suppose the Hausdorff topology* τ_H *on* $C(\Xi)$. *In particular,* (Ξ,τ) *is a metric space and the topology* τ *is induced by a metric* d *on* Ξ. *The Hausdorff metric* H *on* $C(\Xi)$ *is defined, for any distinct* $A,B \in C(\Xi)$ *as*

$$H(\Psi,\tau) = \max\left\{\sup_{a\in\Psi}\left(\inf_{b\in\tau} d(a,b)\right), \sup_{b\in\tau}\left(\inf_{a\in\Psi} d(b,a)\right)\right\}.$$

Then we say here Hausdorff metric H *induces the Hausdorff topology on* $C(\Xi)$.

4.2 Findings

The aim of this study is to introduce some new Theorems in a soft metric space mainly which belongs to generalization of soft ciric and caristi fixed point with the help of convergence criteria we shown distance function in soft metric space approaches to a fixed point and along with give some results, examples and definition to make our work effective and genuine.

4.3 Preliminaries for Soft S-metric space

Let's recall some Properties of Soft set,

Definition 4.3. *If* $\tilde{\aleph}$ *be the mapping from* $\tilde{E} \rightarrow 2^{\tilde{u}}$ *then we say that* $(\zeta,\tilde{E})$ *is a soft set over* $\tilde{\aleph}$. *Which mean that, Soft set over* $\tilde{\aleph}$ *be a parameterized family which include subset of initial universe set* $\tilde{\aleph}$.

Definition 4.4. *Let* $(\zeta_1,\tilde{u})$ *and* $(\zeta_2,\tilde{v})$ *be two soft sets which belongs to* $\tilde{\aleph}$ *has* $(\zeta_3,\tilde{w})$ *where* $w=(u^T v)\forall p\in w, \zeta_3(p)=\zeta_1(p)^T\zeta_2(p)$ *is called the intersection of two soft sets,* $(\zeta_1,\tilde{u})^T(\zeta_2,\tilde{v})=(\zeta_3,\tilde{w})$.

Definition 4.5. *If* $\forall p \in \tilde{u}$, $\zeta(p) = \varphi$ *then we say that* $(\zeta,\tilde{u})$ *is a null soft set on* $\tilde{\aleph}$ *and denoted as* φ.

Definition 4.6. *If* $\forall p \in \tilde{u}, \zeta(p) = u$ *then we say that* $(\zeta, \tilde{u})$ *is an absolute soft set on* $\tilde{\aleph}$ *and denoted as* $\tilde{u}$.

Definition 4.7. *Let* $(\zeta_1, \tilde{u})$, $(\zeta_2, \tilde{v})$ *two soft sets which belongs to* $\tilde{\aleph}$ *has* $(\zeta_3, \tilde{w})$ *where* $w = (u^S v) \forall p \in w$,

$$\zeta_3(p) = \begin{cases} \zeta_1(p) \cap \zeta_2(p) \ldots p \in \tilde{u} \cap \tilde{v} \\ \zeta_1(p) \ldots\ldots p \in \tilde{u} \backslash \tilde{v} \\ \zeta_2(p) \ldots\ldots p \in \tilde{v} \backslash \tilde{u} \end{cases}$$

Is called the union of two soft set and denoted as, $(\zeta_1, \tilde{u})^S (\zeta_2, \tilde{v}) = (\zeta_3, \tilde{w})$.

Definition 4.8. *Let* $(\zeta, \tilde{E})$ *be a soft set on* $\tilde{\aleph}$, $\exists p \in \tilde{E}$ *such that* $\zeta(p) = \{\tilde{v}\}$ *for some* $\tilde{v} \in \tilde{\aleph}$ *and* $\exists q \in \tilde{E} \backslash \{p\}, \zeta(q) = \varphi$. *Then we say* $(\zeta, \tilde{E})$ *is a soft point.*

Definition 4.9. *Let's consider* $\tilde{\aleph}$ *be a soft set and a mapping,* $\tilde{\delta} : sp(\tilde{\aleph}) \times sp(\tilde{\aleph}) \to \mathfrak{R}^*$ *if* $\tilde{\delta}$ *satisfies following four axiom's,*

(Z_1) $\tilde{\delta}(\tilde{u}_{\omega 1}, \tilde{u}_{\omega 2}) \geq 0$ *For every* $\tilde{u}_{\omega 1}, \tilde{u}_{\omega 2} \in \tilde{\aleph}$.

(Z_2) $\tilde{\delta}(\tilde{u}_{\omega 1}, \tilde{u}_{\omega 2}) = 0$ *if and only if* $\tilde{u}_{\omega 1} = \tilde{u}_{\omega 2}$.

(Z_3) $\tilde{\delta}(\tilde{u}_{\omega 1}, \tilde{u}_{\omega 2}) = 0$ *For every* $\tilde{u}_{\omega 1}, \tilde{u}_{\omega 2} \in \tilde{\aleph}$.

(Z_4) $\tilde{\delta}(\tilde{u}_{\omega 1}, \tilde{u}_{\omega 3}) \leq \tilde{\delta}(\tilde{u}_{\omega 1}, \tilde{u}_{\omega 2}) + \tilde{\delta}(\tilde{u}_{\omega 2}, \tilde{u}_{\omega 3})$ *For every* $\tilde{u}_{\omega 1}, \tilde{u}_{\omega 2}, \tilde{u}_{\omega 3} \in \tilde{\aleph}$.

Then soft metric $\tilde{\delta}$ *called as soft metric space and denoted as* $(\tilde{\aleph}, \tilde{\delta}, \rho)$

Definition 4.10. *The sequence* $\{\tilde{u}^n_{\omega_n}\}$ *is said to be a convergent in* $(\tilde{\aleph}, \tilde{\delta}, \rho)$ *if there exist soft point* $\tilde{v}_\sigma \in \tilde{\aleph}$ *such that,* $\lim_{n \to \infty} \tilde{\delta}(\tilde{u}^n_{\omega_n}, \tilde{v}_\sigma) = 0$. *Where* $\{\tilde{u}^n_{\omega_n}\}$ *is a sequence of soft metric space* $(\tilde{\aleph}, \tilde{\delta}, \rho)$.

Definition 4.11. *The sequence* $\{\tilde{u}^n_{\omega_n}\}$ *is said to be a Cauchy sequence in* $(\tilde{\aleph}, \tilde{\delta}, \rho)$ *if there exist soft point* $\tilde{v}_\sigma \in \tilde{\aleph}$ *such that,* $\lim_{i,j \to \infty} \tilde{\delta}(\tilde{u}^i_{\omega i}, \tilde{v}^i_{\sigma j}) = 0$.

Definition 4.12. *If every Cauchy sequence in* $\tilde{\aleph}$ *converges to a some point of* $\tilde{\aleph}$ *then we say that* $(\tilde{\aleph}, \tilde{\delta}, \rho)$ *is complete. Where* $(\tilde{\aleph}, \tilde{\delta}, \rho)$ *is the soft metric space.*

Theorem 4.13. *Define a mapping* $(\zeta,\varphi) : (\tilde{\aleph},\tilde{\delta},\rho) \to (\tilde{\aleph},\tilde{\delta},\rho)$ *Consider there exist,* $(0 \leq \varepsilon < 1)$ *and* $(\tilde{\aleph},\tilde{\delta},\rho)$ *be a complete soft metric space, such that* $\tilde{\delta}((\zeta,\varphi)(\tilde{u}_\omega),(\zeta,\varphi)(\tilde{v}_\sigma)) \leq \varepsilon.\tilde{\delta}(\tilde{u}_\omega,\tilde{v}_\sigma)$ *then inside* $(\tilde{\aleph},\tilde{\delta},\rho)$ *we have* (ζ,φ) *admits unique fixed point.*

Theorem 4.14. *Consider the distance* $\tilde{\delta}$ *and let* $((\tilde{\aleph},\tilde{\delta},\rho),\leq)$ *be a complete soft metric space with partially order on* $\tilde{\aleph}$ *and* $(\zeta,\varphi) : (\tilde{\aleph},\tilde{\delta},\rho) \to (\tilde{\aleph},\tilde{\delta},\rho)$ *be monotonically non- decreasing* $\forall \tilde{u}_\omega,\tilde{v}_\sigma \in \tilde{\aleph}$ *has function,* $\eta : (\tilde{\aleph},\tilde{\delta},\rho) \to [0,+\infty)$ *Satisfying,* $\tilde{\delta}(\tilde{v}_\sigma,(\zeta,\varphi)(\tilde{v}_\sigma)) \leq [\eta(\tilde{v}_\sigma)-\eta((\zeta,\varphi)(\tilde{v}_\sigma))]$*, whenever* $(\zeta,\varphi)(\tilde{v}_\sigma) < \tilde{v}_\sigma$ *for all* $\tilde{v}_\sigma \in \tilde{\aleph}$ *which gives* (ζ,φ) *admits a fixed point iff* $\exists \tilde{v}^0_\sigma \in \aleph \ (\zeta,\phi)(\tilde{v}^0_\sigma) < \tilde{v}^0_\sigma$

In our study, we presented some new soft fixed point theorem with the help of ciric and caristi soft mappings and all the hypothesis with suitable example are discussed.

Example. *consider* $(\tilde{\aleph},\tilde{\delta},\rho) = [0,1] \bigcup [\frac{1+\sqrt{5}}{3},+\infty)$ *having usual distance* $\tilde{\delta}(\tilde{u}_\omega,\tilde{v}_\sigma) = |\tilde{u}_\omega - \tilde{v}_\sigma|$ *given below both function defined on* $(\tilde{\aleph},\tilde{\delta},\rho)$

$$(\zeta,\phi) = \begin{cases} \frac{1}{2\tilde{u}_\omega+1} \ldots\ldots \tilde{u}_\omega \in [\frac{1+\sqrt{5}}{3},+\infty) \\ 1 \ldots\ldots\ldots \tilde{u}_\omega \in [0,1] \end{cases}$$

$$\eta(\tilde{u}_\omega) = \begin{cases} \frac{1}{2} \ldots\ldots\ldots \tilde{u}_\omega \in [0,1) \\ 3 \ldots\ldots \tilde{u}_\omega \in [\frac{1+\sqrt{5}}{3},+\infty) \\ 0 \ldots\ldots\ldots\ldots\ldots \tilde{u}_\omega = 1 \end{cases}$$

We need to prove the following inequality,

$$\tilde{\delta}((\zeta,\varphi)(\tilde{u}_\omega),(\zeta,\varphi)(\tilde{v}_\sigma)) \leq [\eta(\tilde{u}_\omega)-\eta((\zeta,\varphi)(\tilde{u}_\omega))]\max\{1,\vartheta(\tilde{u}_\omega,\tilde{v}_\sigma),\tilde{\delta}(\tilde{u}_\omega,\tilde{v}_\sigma)\}$$

To check hypothesis of above theorem we need to go through following cases,
Case(I). If $\tilde{u}_\omega, \tilde{v}_\sigma \in [\frac{1+\sqrt{5}}{3},+\infty)$ *then our inequality,* $\tilde{\delta}((\zeta,\phi)(\tilde{u}_\omega),(\zeta,\phi)(\tilde{v}_\sigma)) = \frac{2|\tilde{u}_\omega-\tilde{v}_\sigma|}{(1+2\tilde{v}_\sigma)(1+2\tilde{u}_\omega)}$ *and other side must be* $\eta(\tilde{u}_\omega) - \eta((\zeta,\phi)(\tilde{u}_\omega)) = \frac{5}{2}$

$$\tilde{\delta}((\zeta,\varphi)(\tilde{u}_\omega),(\zeta,\varphi)(\tilde{v}_\sigma)) \leq [\eta(\tilde{u}_\omega)-\eta((\zeta,\varphi)(\tilde{u}_\omega))]\max\{1,\vartheta(\tilde{u}_\omega,\tilde{v}_\sigma),\tilde{\delta}(\tilde{u}_\omega,\tilde{v}_\sigma)\}$$

which holds good. Case(II). If $\tilde{u}_\omega, \tilde{v}_\sigma \in [0,1)$ *then our inequality become*

$$\tilde{\delta}((\zeta,\phi)(\tilde{u}_\omega),(\zeta,\phi)(\tilde{v}_\sigma)) = 0 \leq \eta(\tilde{u}_\omega) - \eta((\zeta,\phi)(\tilde{u}_\omega)) = \frac{1}{2}$$

Which gives,

$$\tilde{\delta}((\zeta,\varphi)(\tilde{u}_\omega),(\zeta,\varphi)(\tilde{v}_\sigma)) \leq [\eta(\tilde{u}_\omega)-\eta((\zeta,\varphi)(\tilde{u}_\omega))]\max\{1,\vartheta(\tilde{u}_\omega,\tilde{v}_\sigma),\tilde{\delta}(\tilde{u}_\omega,\tilde{v}_\sigma)\}$$

Case(III).If $\tilde{u}_\omega \in [0,1) \& \tilde{v}_\sigma \in [\frac{1+\sqrt{5}}{3},+\infty)$ *then our inequality become* $\tilde{\delta}((\zeta,\varphi)(\tilde{u}_\omega),(\zeta,\varphi)(\tilde{v}_\sigma)) = 1 - \frac{1}{2\tilde{v}_\sigma+1} \& \eta(\tilde{u}_\omega) - \eta((\zeta,\phi)(\tilde{u}_\omega)) = \frac{5}{2}$ *Which gives,*

$$\tilde{\delta}((\zeta,\varphi)(\tilde{u}_\omega),(\zeta,\varphi)(\tilde{v}_\sigma)) \leq [\eta(\tilde{u}_\omega)-\eta((\zeta,\varphi)(\tilde{u}_\omega))]\max\{1,\vartheta(\tilde{u}_\omega,\tilde{v}_\sigma),\tilde{\delta}(\tilde{u}_\omega,\tilde{v}_\sigma)\}$$

Case(IV). If $\tilde{u}_\omega = 1, \tilde{v}_\sigma \in [\frac{1+\sqrt{5}}{3},+\infty)$ *then our inequality* $\tilde{\delta}((\zeta,\phi)(\tilde{u}_\omega),(\zeta,\phi)(\tilde{v}_\sigma)) = 1 - \frac{1}{2\tilde{u}_\omega+1}$ *and*
other side must be $\eta(\tilde{u}_\omega) - \eta((\zeta,\phi)(\tilde{u}_\omega)) = \frac{5}{2}$

$$\tilde{\delta}((\zeta,\varphi)(\tilde{u}_\omega),(\zeta,\varphi)(\tilde{v}_\sigma)) \leq [\eta(\tilde{u}_\omega)-\eta((\zeta,\varphi)(\tilde{u}_\omega))]\max\{1,\vartheta(\tilde{u}_\omega,\tilde{v}_\sigma),\tilde{\delta}(\tilde{u}_\omega,\tilde{v}_\sigma)\}$$

which holds our inequality. Case(V). If $\tilde{u}_\omega = 1, \tilde{v}_\sigma \in [0,1)$ *then our inequality become,*

$$\tilde{\delta}((\zeta,\phi)(\tilde{u}_\omega),(\zeta,\phi)(\tilde{v}_\sigma)) = 0 \leq \eta(\tilde{u}_\omega) - \eta((\zeta,\phi)(\tilde{u}_\omega)) = \frac{1}{2}$$

$$\tilde{\delta}((\zeta,\varphi)(\tilde{u}_\omega),(\zeta,\varphi)(\tilde{v}_\sigma)) \leq [\eta(\tilde{u}_\omega)-\eta((\zeta,\varphi)(\tilde{u}_\omega))]\max\{1,\vartheta(\tilde{u}_\omega,\tilde{v}_\sigma),\tilde{\delta}(\tilde{u}_\omega,\tilde{v}_\sigma)\}$$

all above cases holds our theorem and (ζ,φ) *gives fixed point as 1.*

Conclusion

In this study, we delved into the world of fixed points and their characteristics within the context of topological spaces. Fixed point theory is a cornerstone of mathematical analysis and topology, offering potent methods for tackling a broad spectrum of issues in both theoretical and applied mathematics. By concentrating on topological spaces, we were able to broaden the scope of classical fixed point results, like those from Banach and Brouwer, into more generalized and abstract scenarios where the usual distance-based assumptions might not apply.

By examining continuous self-maps and leveraging properties such as compactness, connectedness, and other topological features, we pinpointed the conditions that guarantee the existence of fixed points. We showcased how tools like the Tychonoff Fixed Point Theorem and Schauder’s Fixed Point Theorem enable these concepts to be extended to topological vector spaces and compact convex subsets. Additionally, we highlighted that these findings are not just mathematically profound but also incredibly useful in various fields, including economics, game theory, and differential equations.

A key takeaway from our research is the pivotal role that continuity and compactness play in guaranteeing fixed points in environments without a metric structure. We also observed that proving fixed point results in topological spaces often requires more nuanced and abstract reasoning compared to their metric-based counterparts, thereby adding depth to the theory's conceptual framework.

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