A STUDY ON ANALYTIC FUNCTION OF COMPLEX ANALYSIS

Dissertation submitted to the Department of Mathematics in fulfillment of the requirements for the award of the degree of

Master of Science

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CERTIFICATE

This is to certify that **NAZMIN BEGUM CHOUDHURY** bearing **Roll No MAT-09/23** and **Regd. No. MSSV-0023-101-001344** has prepared her dissertation entitled "**A STUDY ON ANALYTIC FUNCTION OF COMPLEX ANALYSIS**" submitted to the Department of Mathematics, **MAHAPURUSHA SRIMANTA SANKARADEVA VISWAVIDYALAYA**, Nagaon, for fulfilment of MSc. degree, under guidance of me and neither the dissertation nor any part thereof has submitted to this or any other university for a research degree or diploma.

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DECLARATION

I, **NAZMIN BEGUM CHOUDHURY**, bearing ROLL No- **MAT-09/23** student of final semester, Department of Mathematics , **MAHAPURUSHA SRIMANTA SANKARADEVA VISHWAVIDYALAYA** (MSSV), do hereby declare that the work incorporated in this dissertation entitled "**A STUDY ON ANALYTIC FUNCTION OF COMPLEX ANALYSIS**", for the award of degree of Master of Science in Mathematics, has been carried out and interpreted by me under the supervision of **DR. MIRA DAS** , Assistant Professor, Department of Mathematics , (MSSV) Nagaon. This dissertation is original and has not been submitted by me for the award of degree of diploma to any other University or Institute. I have faithfully and accurately cited all my sources, including books, journals, handouts and unpublished manuscripts, as well as any other media, such as the internet, letters or significant personal communication.

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ABSTRACT

The study of complex numbers, their derivatives, manipulation, and other properties is known as complex analysis. Complex analysis is a very powerful tool that has a surprising number of practical applications in solving physical problems. Complex analysis is a branch of mathematics that studies functions of complex numbers. It is also known as the theory of functions of a complex variable. Many branches of mathematics, such as algebraic geometry, number theory, analytic combinatorics, and applied mathematics, as well as physics, such as hydrodynamics, thermodynamics, and especially quantum mechanics, benefit from it. Complex analysis has applications in engineering fields such as nuclear, aerospace, mechanical, and electrical engineering by extension.

In this paper we propose a method where imaginary function " $v(x, y)$" is used for a solution of a complex variable where the real part "$u(x, y)$" is unknown. A function is defined and differentiable at All points all points of "D". The paper utilizes the following methods in finding the unknown part "Real part $u(x, y)$" of the analytic function f(z):"The Direct Method, Milne Thompson Method and Exact Differentiable Equation Method". It was found that, out of the three methods used in finding "the real part" of the analytic function, Direct Method is more efficient as it yields results more faster, efficient and accurate.

In chapter 1, I have covered the foundation of analytic functions, including their definition and relationships with other mathematical concepts.

In chapter 2, Here we have discussed Milne Thomson method and to construct analytic function using Milne Thomson method .

In CHAPTER 5, we discussed Direct method of analytic function and solved analytic function using Direct Method.

IV

In CHAPTER 4, the discussion likely covers Exact differential equation method of analytic function and we solved analytic function using this method .

In CHAPTER 5, the discussion likely covers how analytic functions are used to solve problems in complex analysis, physics and engineering.

In CHAPTER-6, Here we discussed about practical uses of analytic function in various field like In Signal processing , In Fluid dynamics, In Computer Graphics etc.

REVIEW OF LITERATURE

Leonhard Euler introduced early ideas of complex functions and established connections between exponential and trigonometric functions through Euler's formula.

Augustin-Louis Cauchy provided the first rigorous definitions and theorems in complex analysis. His work on Cauchy's integral theorem and the Cauchy integral formula laid the groundwork for the analytic function theory.

Bernhard Riemann extended the geometric interpretation of complex functions and developed the theory of Riemann surfaces, crucial for understanding multivalued functions.

Selvaraj, Vasanthi (2010).

In this paper authors introduced the subclass of normalized analytic and univalent functions.The aim of the present paper is to obtain sharp upper bounds for the functional $|a_2a_4 - a_3^2|$for functions which is normalized analytic and univalent functions.

Selvaraj, Vasanthi (2011).

In this paper authors introduced new subclasses of convex and starlike functions with respect to other points. The main result of this paper is to obtain the coefficient inequalities for the classes M_s(α, A, B) and M_c(α, A, B)

Murugusundaramoorthy, Sivasubramanian, Raina (2009).

In this paper authors introduced a new subclass UH (α β γ δ λ κ) of functions which are analytic in the open disk. Also they studied various results include the coefficient estimates and distortion bounds, radii of close to convexity, starlikeness and convexity and integral mean inqualities for functions belonging to the same class and briefly indicated the relevances of the main results.

Reddy,Reddy (2007)

The aim of this present paper authors introduced the class S P_s(β) which is a subclass of SP_s (β) (0<∞). Also they study neighbourhoods of the class and proved a necessary and sufficient condition in term of convolutions for a function f to be in SP_s (β) and also showed that the class SPS(β) is closed under convolution with functions f which are convex univalent in E

Lee,Khairnar,Rajas (2011).

In this paper authors introduced new class of p valentuniformaly convex functions by using differential operator and obtained the coefficient bounds,extreme bounds and radius of starlikeness for the functions belongaing to this generalized class.Furthermore partial sums f_k(z) of function f(z) in the class s* (λ α β) are considered and obtained various sharp results.

Aouf, Magesh, Jothibasu, Murthy (2013).

The aim of the present paper to study a new subclass of meromorphic univalent functions defined by convolution structure.Also investigated various characteristics properties and obtained partial sum of the class.

Ghanim, Darus (2009).

In this paper authors defined a new subclass of uniformalystarlike and convex functions with negative coefficient estimates such as distortion bounds, closure theorems and extreme points for functions belonging to the new class.

Amer, Darus(2012).

The aim of the present investigation is to obtain coefficient bounds and extreme points of the subclass of starlike functions associated with wright's generalized hypergometric functions. Also they study radius of convexity and closer properties for functions in the same generalized class.

Janteng, Halim, Darus (2007).

The object of this paper is to obtaining sharp upper bounds for the functional $|a_2a_4 - a_3^2|$ for the normalized analytic univalent functions belonging to starlike and convex functions.

Juma, Kulkarni (2007).

In this paper authors introduced a class of meromorphic multivalent functions by using linear operator associated with differential operator. They have obtained some interesting results with derived the coefficient bounds and coefficient inequalities.

Jamal, Mousa (2006).

In this paper authors introduced a new class of uniformly convex functions defined by using a certain linear operator. Coefficient estimates, distortion theorems, and other Interesting properties of this class of functions are studied. Further class preserving Integral operator and some closed theorems for this class are also indicated.

Introduction

Complex analysis is a branch of mathematics that involves functions of complex numbers. It provides an extremely powerful tool with an unexpectedly large number of applications, including in number theory, applied mathematics, physics, hydrodynamics, thermodynamics, and electrical engineering. Rapid growth in the theory of complex analysis and in its applications has resulted in continued interest in its study by students in many disciplines. This has given complex analysis a distinct place in mathematics curricula all over the world, and it is now being taught at various levels in almost every institution.

Although several excellent books on complex analysis have been written, the present rigorous and perspicuous introductory text can be used directly in class for students of applied sciences. In fact, in an effort to bring the subject to a wider audience, we provide a compact, but thorough, introduction to the subject in An Introduction to Complex Analysis. This book is intended for readers who have had a course in calculus, and hence it can be used for a senior undergraduate course. It should also be suitable for a beginning graduate course because in undergraduate courses students do not have any exposure to various intricate concepts, perhaps due to an inadequate level of mathematical sophistication.

All theorems and their proofs, and the presentation is rather unconventional. It comprises 50 class tested lectures that we have given mostly to math majors and engineering students at various institutions all over the globe over a period of almost 40 years. These lectures provide flexibility in the choice of material for a particular one-semester course. It is our belief that the content in a particular lecture, together with the problems therein, provides fairly adequate coverage of the topic under study.

All the central topics in the undergraduate mathematics syllabus, complex analysis is arguably the most attractive. The huge consequences emanating from the assumption of differentiability, and the sheer power of the methods deriving from Cauchy's Theorem never fail to impress, and undergraduates actively enjoy exploring the applications of the Residue Theorem.

Complex analysis is not an elementary topic, and one of the problems facing lecturers is that many of their students, particularly those with an "applied" orientation, approach the topic with little or no familiarity with the E-8 arguments that are at the core of a serious course in analysis. It is, however, possible to appreciate the essence of complex analysis without delving too deeply into the fine detail of the proofs, and in the earlier part of the book I have starred some of the more technical proofs that may safely be omitted. Proofs ' are, however, given, since the development of more advanced analytical skills comes from imitating the techniques used in proving the major results.

The study of complex numbers, their derivatives, manipulation, and other properties is known as complex analysis. Complex analysis is a very powerful tool that has a surprising number of practical applications in solving physical problems. Complex analysis is a branch of mathematics that studies functions of complex numbers. It is also known as the theory of functions of a complex variable. Many branches of mathematics, such as algebraic geometry, number theory, analytic combinatorics, and applied mathematics, as well as physics, such as hydrodynamics, thermodynamics, and especially quantum mechanics, benefit from it. Complex analysis has applications in engineering fields such as nuclear, aerospace, mechanical, and electrical engineering by extension

The solution of certain algebraic equations by the Italian mathematicians GirolamoCardano and Raphael Bombelli in the 16th century provided the first indications that complex numbers might be useful. They were fully established as sensible mathematical concepts by the 18th century, after a long and contentious history. They remained on the mathematical periphery until it was discovered that analysis can be applied to the complex domain as well. The result was such a powerful addition to the mathematical toolbox that philosophical questions about the meaning of complex numbers were lost in the rushto use them. Soon, the mathematical community had grown so accustomed to complex numbers that it was difficult to remember that there had ever been a philosophical issue.

Complex analysis is a branch of mathematics that investigates the analytical properties of complex variable functions. It sits at the crossroads of several branches of mathematics, both pure and applied, and has ties to asymptotic, harmonic, and numerical analysis. Complex variable techniques are extremely powerful, with a wide range of applications in the solution of physical problems. Solution methods for free-boundary problems such as Hele-Shaw and Stokes flow, conformal mappings, Fourier and other transform methods, and Riemann-Hilbert problems are all covered by this discipline. Due to a number of special properties of the complex domain, many problems that are difficult to solve in the real domain can be solved more easily when transformed into complex variables. Complex numbers are defined as the set of all numbers $z = x + yi$, where x and y are real numbers. We denote the set of all complex numbers by C. (On the blackboard we will usually write C –this font is called blackboard bold.) We call x the real part of z. This is denoted by $x = Re(z)$. We call y the imaginary part of z. This is denoted by

$y = Im(z)$.

CHAPTER-1

PRELIMINARIES

1.1Introduction: In this chapter we will discuss about basic definitions of complex analysis such as complex variable, complex number, complex plane, complex functions, continuity etc.

1.2 Basic definitions

Definition Complex variable: The quantity z = x + iy where x and y are two independent real variables is called a complex variable .The Argand plane in which the variable z is represented by the points is called z plane.

Definition Complex number : A complex number is the sum of a real number and an imaginary number**.** A complex number is of the form a + ib and is usually represented by z. Here both a and b are real numbers. The value 'a' is called the real part which is denoted by Re (z), and 'b' is called the imaginary part Im (z).

Definition Complex function: Complex variable functions or complex functions are functions that assign complex numbers for complex numbers**.** Let C be the set of complex numbers. A function f : C → C is a rule which associates with z ∈ C, a unique w ∈ C, written as w = f (z). Here, z = x + iy .

1.2.1 Limit of a Function of a Complex Variable

Let $w = f(z)$ be any single valued function defined in the deleted neighbourhood of $z = a$. We say that $f(z)$ tends to limit l as z tends to a along any path in a defined region, if to each positive arbitrary number ∈, however small there corresponds a positive number depending upon ∈ such that $|f(z) - l)| < \in$ for all points of the region for which $0 < |z - a| < \partial$

symbolically we write ,

$\lim_{z \to a} f(z) = f(a)$.

Definition Continuity: A function of a complex variable is said to be continuous in a region R if it is continuous at each point in R. If a function f is not continuous at a point z 0 then we say that f is discontinuous at z 0. For example, the function f (z) = 1 1 + z 2 is discontinuous at z = i and z = − i.

Definition Neighbourhood of a point : An ε neighborhood, also called open ball or open disk, of a complex number z_0 consists of all points z lying inside but not on a circle centred at z_0 and with radius ε> 0 and is expressed by (1) B— ε (z_0) = {z: | $z - z_0$| <ε}. (2) B — ε (z_0) = {z: | $z - z_0$ | ≤ ε}.

Definition Deleted Neighbourhood of z_0: A delta or **δ** neighborhood of a point z_0 is the set of all points **z** such that $|z - z_0| < \delta$ where δ is any given positive (real) number. A point z_0 is called a limit point, cluster point or a point of accumulation of a point set S if every deleted δ neighborhood of z_0 contains points of S.

Definition Differentiability: A complex function f (z) is differentiable at a point $z_0 \in$ C if and only if the following limit difference quotient exists (1) f ′ (z_0) = lim $z \to z_0$ f (z) − f (z_0) $z - z_0$. Alternatively, letting Δ z = $z - z_0$ we can write (2) f ′ (z_0) = lim Δ z → 0 f (z_0 +Δz) − f (z_0) Δz

Definition singularity**,** of a function of the complex variable *z* is a point at which it is not analytic.

Definition Analytic function: A complex function is called analytic at a point z_0 in its domain if it is differentiable in a neighbourhood of the point z_0.

A function f(z) is said to be analytic in a region R of the z-plane if it is

analytic at every point of R.

The terms regular and holomorphic are also sometimes used as synonyms for

Analytic.

Definition Entire function: A function is entire if it is analytic everywhere on the complex plane. Example include exponential , sine, cosine and hyperbolic function.

Definition Holomorphic function:_ A function which is analytic , it is also sometimes called holomorphic function.

1.2.2 Single Valued and Many Valued Function:

If for every point z in a region R of the z - plane there corresponds a unique value for w, then w is called a single valued function of z.

If more than one value of w corresponds to a point z in a region R of the z - Plane, then w is said to be a many (multiple) valued function of z in that region.

For example: w = f(z) = z is a single valued function of z, but w = f(z) = √ z is a many valued function of z

1.3 Properties of analytic function

- The sums, products, and compositions of analytic functions are analytic.
- The reciprocal of an analytic function that is nowhere zero is analytic, as is the inverse of an invertible analytic function whose derivative is nowhere zero.
- Any analytic function is smooth, that is, infinitely differentiable. The converse is not true for real functions; in fact, in a certain sense, the real analytic functions are sparse compared to all real infinitely differentiable functions. For the complex numbers, the converse does hold, and in fact any function differentiable *once* on an open set is analytic on that set.

For any open set $\Omega \leq c$, the set $A(\Omega)$ of all analytic functions is a $\mu : \Omega \rightarrow$ c is a Fréchet space with respect to the uniform convergence on compact sets. The fact that uniform limits on compact sets of analytic functions are analytic is an easy consequence of Morera's theorem. The set $A_{\infty}(\Omega)$ of all bounded analytic functions with the supremum norm is a Banach space.

A polynomial cannot be zero at too many points unless it is the zero polynomial (more precisely, the number of zeros is at most the degree of the polynomial). A similar but weaker statement holds for analytic functions. If the set of zeros of an analytic function f has an accumulation point inside its domain, then f is zero everywhere on the connected component containing the accumulation point. In other words, if (r_n) is a sequence of distinct numbers such that $f(r_n) = 0$ for all n and this sequence converges to a point r in the domain of D, then f is identically zero on the connected component of D containing r. This is known as the identity theorem.

Also, if all the derivatives of an analytic function at a point are zero, the function is constant on the corresponding connected component.

These statements imply that while analytic functions do have more degrees of freedom than polynomials , they are still quite rigid .

1.4 FUNCTIONS OF THE COMPLEX PLANE:

Continuous functions

Let f be a function defined on a set Ω of complex numbers. We say that f is continuous at the point $z_0 \in \Omega$ if for every $\in > 0$ there exists $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$ then $|f(z - f(z_0)| < \in$. An equivalent definition is that for every sequence $\{z_1, z_2, ...\} \subset \Omega$ such that $\lim z_n = z_0$, then $\lim f(z_n) = f(z_0)$. The function f is said to be continuous on Ω if it is continuous at every point of Ω. Sums and products of continuous functions are also continuous.

Since the notions of convergence for complex numbers and points in R2 are the same, the function f of the complex argument z = x + iy is continuous if and only if it is continuous viewed as a function of the two real variables x and y.

By the triangle inequality, it is immediate that if f is continuous, then the real-valued function defined by $z \rightarrow |f(z)|$ is continuous. We say that f attains a maximum at the point $z_0 \in \Omega$ if $|\mathrm{f}f(z)| \leq |f(z_0)|$ for all $z \in \Omega$, with the inequality reversed for the definition of a minimum.

Holomorphic functions

We now present a notion that is central to complex analysis, and in distinction to our previous discussion we introduce a definition that is genuinely complex in nature.

Let Ω be an open set in C and f a complex-valued function on Ω. The function f is holomorphic at the point $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0+h)-f(z_0)}{h} \qquad (1)$$

Converges to a limit when $h \rightarrow 0$. Here $h \in$ C and $h \neq 0$ 0 with $z_0 + h \in \Omega$, so that the quotient is well defined. The limit of the quotient, when it exists, is denoted by $f^{/}(z_0)$, and is called the derivative of f at z_0 :

$$f^{/}(z_0) = \lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

It should be emphasized that in the above limit, h is a complex number that may approach 0 from any direction.

The function f is said to be holomorphic on Ω if f is holomorphic at every point of Ω. If c is a closed subset of C, we say that f is holomorphic on C if f is holomorphic in some open set containing c. Finally, if f is holomorphic in all of C we say that f is entire.

1.5 Derivative of a Complex Function

A function $f(z)$ is said to be differentiable at z_0 if $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. This limit is called the derivative of $f(z)$ at z_0 and it is denoted as $f^{/}(z)$ on putting $z - z_0 = \Delta z$, we have,

$$f^{/}(z) = \text{if } \lim_{z \to z_0} \frac{f^{/}(z_0+\Delta z)-f(z_0)}{\Delta z} \text{ when limit exists.}$$

The function $f(z)$ is said to be differentiable at z if limit exists.

CHAPTER 2

MILNE THOMSON METHOD

2.1 Introduction: In mathematics, the Milne-Thomson method is a method for finding a holomorphic function whose real or imaginary part is given. It is named after Louis Melville Milne-Thomson**.**

2.2 To construct an Analytic function by Milne Thomson method

A complex Variable function given by $f(z) = U(x, y) + iV(x, y)$ is said to be analytic if it satisfies the following conditions:

Case 1: If the real part u of f(z) is given ,

Step 1: Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

Step 2: Find $\frac{\partial f(z)}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}$

$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (by C-R equation)

Step 3: put $x = z$ and $y = 0$ in $\frac{\partial f(z)}{\partial x}$.

Step 4: Integrate $\frac{\partial f(z)}{\partial x}$ to obtain $f(z)$.

Case 2:_ If the imaginary part v of $f(z)$ is given-

Step1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

Step 2: Find $\frac{\partial f(z)}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$ (by C-R equation)

Step 3: put $x = z$ and $y = 0$ in $\frac{\partial f(z)}{\partial x}$.

Step 4: Integrate $\frac{\partial f(z)}{\partial x}$ to obtain $f(z)$.

2.3 PROBLEMS

Problem:1 Construct the analytic function $(\boldsymbol{z})$ for which the real part is $e^x cosy$.

Solution: Given $u = e^x cosy$

$=> u_x = e^x cosy$ $[\therefore cos0 = 1]$

$=> u_x(z,0) = e^x$

$=> u_y = e^x cosy \quad [sin0 = 0]$

$=> u_y(z,0) = 0$

$\therefore f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + c$, where c is a complex constant.

$\therefore f(z) = \int e^z dz - i\int 0dz + c$

$= e^z + c$.

Problem:2 Construct the analytic function where real part is

$$u = x^3 - 3xy^2 + 3x^2 - \ 3y^2 + 1$$

Solution: Given, $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$u_x = 3x^2 - 3y^2 + 6x$$

$$=> u_x(z,0) = 3z^2 - 0 + 6z$$

$$u_y = 0 - 6xy + 0 - 6y$$

$$=> u_y(z,0) = 0$$

$f(z) = \int u_x\,(z,0)dz - i\int u_y\,(z,0)dz + c$,[by Milne Thomson rule]

(where c is a complex constant.)

$f(z) = \int(3z^2 + 6z)dz - i\int 0 + dz + c$

$= 3\frac{x^3}{3} + 6\frac{z^2}{2} + c = z^3 + 3z^2 + c.$

Problem:3 Construct the analytic function $w = u + iv$ if $u = e^{2x}(xcos2y - ysin2y)$.

Solution Given , $u = e^{2x}(xcos2y - ysin2y)$.

$$u_x = e^{2x}[cos2y] + (xcos2y - ysin2y)[2e^{2x}]$$

$u_x(z,0) = e^{2z}[1] + [z(1) - 0][2e^{2z}]$

$$= e^{2z} + 2ze^{2z}$$

$$= (1 + 2z)\ e^{2x}$$

$$u_y = e^{2x}[-2xsiny - (y2cos2y + sin2y)]$$

$$u_y(z,0) = e^{2z}[-0(0+0)] = 0$$

$\therefore f(z) = \int u_x\,(z,0)dz - i\int u_y\,(z,0)dz + c\ [by\ Milne - Thomson\ rule],$

where c is a constant .

$$f(z) = \int(1+2z)e^{2z}\,dz - i\int 0 + dz + c$$

$$= \int(1+2z)e^{2z}\,dz + c$$

$$= (1+2z)\frac{e^{2z}}{2} - 2\frac{e^{2z}}{4} + c\ [since\ \int uvdz = uv_1 - u^{/}\,v_2 + u^{/}v_3 \ldots.\quad]$$

$$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + c$$

$$= ze^{2z} + c\ .$$

Problem:4 If $f(z) = u + iv$ is an analytic function $u - v = e^x(cosy - siny)$, find $f(z)$ in terms of z .

<u>Solution</u> : Given ,

$$u - v = e^x(cosy - siny)\ \ldots\ldots\ldots\ldots\text{(A)}$$

Differentiate (A) w.r to x we get

$$u_x - v_x = e^x(cosy - siny)$$

$$u_x(z,0) - v_x(z,0) = e^z\ \ldots\ldots\ldots\ldots\text{(1)}$$

Differentiate (A) w.r to y we get

$$u_y - v_y = e^x(-siny - cosy)$$

$$u_y(z,0) - v_y(z,0) = e^z[-1]$$

i.e $u_y(z,0) - v_y(z,0) = -\,e^z$

$-\,v_x(z,0) -\ u_x(z,0) =\ -\,e^z$ ……….(2) [by C-R conditions]

$\therefore$ (1)+(2) $=> -2\,v_x(z,0) = 0$

$$=> v_x(z,0) = 0$$

(1) $=>\ u_x(z,0) =\ e^z$

$f(z) = \int u_x\,(z,0)dz + i\int v_x(z,0)dz + c$ [by Milne Thomson rule]

$$f(z) =\ \int e^z dz + i0 + c$$

$$= e^z + c\ .$$

CHAPTER 3

DIRECT METHOD

3.1 Introduction: The "direct method" in complex analysis is a technique used to find an analytic function when the real or imaginary part of the function is known. It involves finding the partial derivatives of the given part with respect to x and y, then solving the resulting equations to find the unknown part.

3.2 Direct method: To find the real part of an analytic function where the imaginary part "$v(x,y)$ " is given, we apply the following steps:

A function $f(z)$is said to be analytic function if :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Step 1: Find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

Step 2: From $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ we have U=$\int \frac{\partial v}{\partial y} dx$

Step 3: From $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ we have U=$\int -\frac{\partial v}{\partial x} dy$

Step 4: add step 2 and step 3 by adding both similar and non similar terms

Step 5: produce $f(z) = u(x,y) + iv(x,y)$.

3.3 PROBLEMS OF ANALYTIC FUNCTION USING DIRECT METHOD

Problem :1 Find the analytic function whose imaginary part is given by

$v(x,y) = 3x^2y - y^3 + 6xy$.

Solution: From the C-R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ………(1)

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ……(2)

Since $v(x,y) = 3x^2y - y^3 + 6xy$

$$\frac{\partial v}{\partial x} = 6xy + 6y$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x$$

Now from (1) $U = \int \frac{\partial v}{\partial y} dx$

$$U = \int (3x^2 - 3y^2 + 6x) dx$$

$$U = x^3 - 3y^2x + 3x^2 + c \text{..........(3)}$$

Again, we have $U = \int -\frac{\partial v}{\partial x} dy$

$$U = \int -(6x^2 + 6y) dy$$

$$U = -3xy^2 - 3y^2 + c \text{.........(4)}$$

Adding (3) and (4) by adding both similar and non similar terms we get

$U(x, y) = x^3 - 3y^2x + 3x^2 + c$

$U(x, y) = -3xy^2 - 3y^2 + c$

Similar terms : $-3xy^2, c$

Non similar terms : $x^3, -3y^2, 3x^2$

$U(x, y) = x^3 - 3y^2x + 3x^2 - 3y^2 + c$

$f(z) = (x^3 - 3y^2x + 3x^2 - 3y^2 + c) + i(3x^2y - y^3 + 6xy)$.

Problem:2 Find the analytic function whose imaginary part is given by

$$v(x, y) = 3x^2y - y^3 .$$

Solution Given,

$$v(x, y) = 3x^2y - y^3$$

$$\frac{\partial v}{\partial x} = 6xy , \frac{\partial v}{\partial y} = 3x^2 - y^2$$

$$U = \int \frac{\partial v}{\partial y} dx => U = \int 3x^2 - 3y^2 dx$$

$$=> U = x^3 - 3y^3x + c \text{ ....(1)}$$

Since $U = \int -\frac{\partial v}{\partial x} dy$

$$= \int -6xy dy$$

$$=> U = -3xy^2 + c \text{ ....(2)}$$

Add step (1) and (2) by adding both similar and non similar terms

$U(x,y) = x^3 - 3y^2x + c$

$U(x,y) = -3xy^2 + c$

Similar terms : $-3xy^2, c$

Non similar terms : x^3

$U(x,y) = x^3 - 3y^2x + c$

$\therefore\ f(z) = x^3 - 3y^2x + c + i(3x^2y - y^3).$

CHAPTER 4

EXACT DIFFERENTIAL EQUATION METHOD:

4.1 Introduction: In complex analysis, the method of exact differential equations helps solve first-order nonlinear differential equations by utilizing the relationship between partial derivatives. An exact differential equation is one where a function exists whose total derivative equals the given equation. This method is particularly useful in situations where the equation can be expressed in a specific form, and the partial derivatives of the coefficients satisfy a certain condition.

4.2 Exact differential equation method :

A complex Variable function given by $f(z) = U(x,y) + iV(x,y)$ is said to be analytic if it satisfies the following conditions:

Step 1 : Find $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Step 2 : $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}$ we have

$$U = \int \frac{\partial v}{\partial y} dx + g(y)$$

Step 3 : $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[\int \frac{\partial v}{\partial y} dx + g(y)]$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int \frac{\partial v}{\partial y} dx\right] + g^{/}(y)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int \frac{\partial v}{\partial y} dx\right] + g^{/}(y) = -\frac{\partial v}{\partial x}$$

Step 4 :

$$g^{/}(y) = -\frac{\partial v}{\partial x} - \frac{\partial}{\partial y}\left[\int \frac{\partial v}{\partial y} dx\right]$$

$$g^{/}(y) = -\{\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\left[\int \frac{\partial v}{\partial y} dx\right]\}$$

Step 5 : find $\int g^{/}(y)$

Step 6 : Substitute step 5 in step 2 .

4.3 PROBLEMS

Problem:1 Find the analytic function whose imaginary part is given by

$$v(x, y) = 3x^2y - y^3 + 6xy \text{ .}$$

Solution: from the analytic function

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ………(1) and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ……..(2)

And by applying step 1 and step 5 we have

Step 1 : find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

$$v(x, y) = 3x^2y - y^3 + 6xy$$

$$\frac{\partial v}{\partial x} = 6xy + 6y$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x$$

Step 2 : from (1) $U = \int \frac{\partial v}{\partial y} dx + g(y)$

$$U = \int (3x^2 - 3y^2 + 6x) dx + g(y)$$

$$U = x^3 - 3y^2x + 3x^2 + g(y)$$

Step 3 : $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial y} = -6xy + g^{/}(y) = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial y} = -6xy + g^{/}(y) = -6xy - 6y$$

Step 4 : $g^{/}(y) = -6xy - 6y + 6xy$

$$g^{/}(y) = -6y$$

Step 5 : $\int g^{/}(y) = \int -6y$

$$g(y) = -3y^2$$

Step 6 : substitute step 5 in step 2

$$U(x, y) = x^3 - 3y^2x + 3x^2 - 3y^2$$

$f(z) = (x^3 - 3y^2x + 3x^2 - 3y^2) + i(3x^2y - y^3 + 6xy)$.

Problem:2. Find the analytic function whose imaginary part is given by $v(x, y) = cosxsinhy$.

Solution:

Step 1 : find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

$$v(x, y) = cosxsinhy$$

$$\frac{\partial v}{\partial x} = sinxsinhy$$

$$\frac{\partial v}{\partial y} = cosxcoshy$$

Step 2 : from (1) $U = \int \frac{\partial v}{\partial y} dx + g(y)$

$$U = \int cosxcoshydx + g(y)$$

$$U = sinxcoshy + g(y)$$

Step 3 : $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial y} = sinxsinhy + g^{/}(y) = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial y} = sinxsinhy + g^{/}(y) = -(-sinxsinhy)$$

$$g^{/}(y) = sinxsinhy - sinxsinhy$$

$$g^{/}(y) = 0$$

Step 4 : $g^{/}(y) = 0$

Step 5 : $\int g^{/}(y) = \int 0$

$$g(y) = c$$

Step 6 : substitute step 5 in step 2

$$U(x, y) = sinxcoshy + c$$

$$\therefore f(z) = (sinxcoshy + c) + i(cosxsinhy).$$

CHAPTER 5

CAUCHY-RIEMANN EQUATIONS

5.1 Introduction: The Cauchy-Riemann equations are a fundamental part of complex analysis. They provide a necessary condition for a complex function to be differentiable (and hence analytic or holomorphic).

For the function

$f(z) = u(x, y) + iv(x, y)$ to be differentiable at a point $z_0 = x + iy$, the partial derivatives of u and v must satisfy the Cauchy- Reimann equations at that point :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \; .$$

5.2 Derivation of Cauchy Rimann equations

Statement: If $f(z) = u(x, y) + iv(x, y)$ is analytic in a region R of the z- plane then

i) u_x , u_y, v_x, v_y exist and

ii) $u_x = v_y$ and $u_y = -v_x$ at every point in that region.

Proof : Let $f(z) = u(x, y) + iv(x, y)$

We first assume that $f(z)$ is analytic in a region R . Then by the definition $f(z)$ has a derivative $f'(z)$ everywhere in R .

Now

$$f'(z) = \lim_{z \to z_0} \frac{f'(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Let $z = x + iy$

$\Delta z = \Delta x + i\Delta y$

$(z + \Delta z) = (x + \Delta x) + i(y + \Delta y)$

$\therefore \; f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + i\, v(x + \Delta x, y + \Delta y)$

We know that $f(z) = u(x, y) + iv(x, y)$

Now

$$f'(z) = \lim_{(\Delta x + \Delta y) \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [(u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

Case (i) If $\Delta z \to 0$, first we assume that $\Delta y = 0$ and $\Delta x \to 0$

$$\therefore\ f^{/}(z) = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) + iv(x+\Delta x, y)] - [(u(x,y) + iv(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) - u(x,y)] + i[v(x+\Delta x, y)] - v(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) - u(x,y)]]}{\Delta x} + i \lim_{\Delta x \to 0} \frac{[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$\therefore\ f^{/}(z) = u_x + iv_x$ …………………(1)

Case(ii) If $\Delta z \to 0$, first we assume that $\Delta x = 0$ and $\Delta y \to 0$

$$\therefore\ f^{/}(z) = \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) + iv(x, y+\Delta y)] - [(u(x,y) + iv(x,y)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) - u(x,y)] + i[v(x, y+\Delta y)] - v(x,y)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) - u(x,y)]]}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{[v(x, y+\Delta y) - v(x,y)]}{i\Delta y}$$

$$= \frac{\partial v}{\partial y} + \frac{1}{i}\frac{\partial u}{\partial y}$$

$\therefore\ f^{/}(z) = -iu_y + v_y$ ………………..(2) (since $1/_i = -i$)

from (1) and (2), we get

$u_x + iv_x = -iu_y + v_y$

Equating real and imaginary parts we get ,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

The equations are called Cauchy –Reimann equations or C-R Equations .

Therefore the function $f(z)$ to be analytic at the point z , it is necessary that the four partial derivatives u_x, u_y , v_x and v_y should exist and satisfy the C-R equations.

5.3 Sufficient condition for f(z) to be analytic :

Statement : The singled valued continuous function f(z) = u + i v is analytic in a region R of the z-plane, if the four partial derivatives u_x, u_y , v_x and v_y (i) exist , (ii) continuous , (iii) they satisfy the C-R equations $u_x = v_y$ and $u_y = -v_x$ at every point of R.

Note:- All polynomials, trigonometric, exponential functions are continuous .

5.4 Cauchy-Reimann Equations in polar form :

Statement : If $f(z) = u(r, \theta) + v(r, \theta)$ is differential at $z = re^{i\theta}$, then

$$\frac{\partial u}{\partial r} = \left(\frac{1}{r}\right)\frac{\partial v}{\partial \theta} \Rightarrow u_r = \left(\frac{1}{r}\right) v_\theta$$

$$\frac{\partial v}{\partial r} = -\left(\frac{1}{r}\right)\frac{\partial u}{\partial \theta} \Rightarrow v_r = -\left(\frac{1}{r}\right) u_\theta$$

Proof: Let $z = re^{i\theta}$

$$f(z) = u + iv$$

i.e., $u + iv = f(re^{i\theta})$

Differentiating partially w.r.t 'r' we get

$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f^{/}(re^{i\theta})\, e^{i\theta}$ ………..(2)

Differentiating partially w.r.t 'θ' we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f^{/}(re^{i\theta})(re^{i\theta})(i)$$

$$= (ri)f^{/}(re^{i\theta})(e^{i\theta})$$

$$= (ri)\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \quad \text{from equation (2)}$$

$$= ir\left(\frac{\partial u}{\partial r}\right) - r\left(\frac{\partial v}{\partial r}\right) \longrightarrow (3)$$

Equating real and imaginary parts in equauation (3) , we get ,

$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$

i.e., $u_\theta = -rv$, and $v_\theta = ru$,

(or) $v_r = \left(\frac{-1}{r}\right)u_\theta$ and $u_r = \left(\frac{1}{r}\right) v_\theta$.

EXAMPLES :

1) Show that $f(z) = z^3$ is analytic .

Proof : Given $f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2iy + 3x(iy)^2 + (iy)^3$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

We know that $f(z) = u + iv$

So, $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$, $\frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial x} = -6xy$, $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$

From the above equations we get ,

$u_x = v_y$ and $u_y = -v_x$

∴ C-R equations are satisfied .

Here u_x, u_y, v_x and v_y exists and continuous .

Hence the given function is analytic .

2) Examine the analyticity of the following functions and its derivatives .

i) $f(z) = e^z$

ii) $f(z) = cosz$

Solution : i) $f(z) = e^z = e^{x+iy} = e^x(cosy + isiny)$

Here $u = e^x cosy$ and $v = e^x siny$

$u_x = e^x cosy$ $v_x = e^x siny$

$$u_y = -e^x siny \qquad v_y = e^x cosy$$

$\therefore u_x = v_y$ and $u_y = -v_x$

Hence C-R equations are satisfied.

$f(z) is\ analytic\ everywhere\ in\ the\ complex\ plane$.

Now $f^{/}(z) = u_x + i\ v_x$

$$= e^x cosy + ie^x siny$$

$$= e^x(cosy + isiny)$$

$$= e^x e^{iy}$$

$$= e^{x+iy}$$

$$= e^z$$

ii) Solution :

$$f(z) = cosz = \cos(x + iy)$$

$$= cosxcos(iy) - sinxsin(iy)$$

$$= cosxcoshy - isinxsinhy \quad (\text{Q}\cos(ix) = coshx$$

$$\sin(ix) = isinhx$$

$\therefore\ u = cosxcoshy \qquad v = -isinxsinhy$

$u_x = -sinxcoshy \qquad v_x = -cosxsinhy$

$u_y = cosxsinhy \qquad v_y = -sinxcoshy$

$\therefore\ u_x = v_y$ and $u_y = -v_x$

Hence C-R equations satisfied

$\therefore$ It is analytic.

Also $f^{/}(z) = u_x + i\ v_x$

$$= -sinxcoshy + i\,(-cosxsinhy)$$

$$= -sinxcosiy + i\left(-cosx\left(\frac{1}{i}\right)\sin(iy)\right)$$

$$= -[\sin(x + iy)]$$

$$= -sinz\ .$$

3. Show that $f(z) = logz$ is analytic everywhere is analytic except at the origin and find its derivative .

<u>Solution :</u> $f(z) = \log z$

$$= \log\ (re^{i\theta}) \quad \text{since}(z = re^{i\theta})$$

$$= \log r + loge^{i\theta}$$

$$= \log r + ie^{\theta}$$

We know that $f(z) = u + iv)$

Here $u = \log r \quad v = \theta$

$\therefore\ u_r = \frac{1}{r} \quad v_r = 0$

$u_\theta = 0 \qquad v_\theta = 1$

$\therefore\ u_r\ ,\ v_r\ , u_\theta\ ,\ v_\theta$ exist are continuous and satisfy C-R equations

$u_r = \left(\frac{1}{r}\right) v_\theta$ and $v_r = -\left(\frac{1}{r}\right) u_\theta$ everywhere except at $z = 0$.

5.5 Harmonic functions :

A real valued function of two real variables x and y is said to be harmonic, if

i) The second order partial derivatives u_{xx} , u_{xy} , u_{yx} , u_{yy} exist and they are continuous.

and

ii) The Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ satisfies.

5.6 Conjugate Harmonic functions:

If **u**+**iv** is an analytic function of **z** then **v** is called a conjugate harmonic function of **u**; (**or**) **u** is called a conjugate harmonic function of **v**; (or) u and v are called conjugate harmonic functions.

Method to find out the Harmonic conjugate :

Let $f(z) = u + iv$ be an analytic function .

Given : $\boldsymbol{u(x,y)}$

$\therefore \boldsymbol{v = \int -u_y dx + \int u_x dy}$.

- If $\boldsymbol{f(z) = e^x(cosy + isiny)}$ is analytic function prove that u, v are harmonic functions .

  Solution : To prove that u and v are harmonic functions

  $\boldsymbol{i.e,}$ $\boldsymbol{u_{xx} + u_{yy}}= 0$ $\boldsymbol{v_{xx} + v_{yy} = 0}$

  Here $\boldsymbol{u = e^x cosy}$ and $\boldsymbol{v = e^x siny}$

  $\boldsymbol{u_x = e^x cosy}$ $\boldsymbol{v_x = e^x siny}$

  $\boldsymbol{u_{xx} = e^x cosy}$ $\boldsymbol{v_{xx} = e^x siny}$

  $\boldsymbol{u_y = -e^x siny}$ $\boldsymbol{v_y = e^x cosy}$

  $\boldsymbol{u_{yy} = -e^x cosy}$ $\boldsymbol{v_{yy} = -e^x siny}$

  $\therefore \boldsymbol{u_{xx} + u_{yy} = e^x cosy - e^x cosy = 0}$

  and $\boldsymbol{v_{xx} + v_{yy} = e^x siny - e^x siny = 0}$

  $\therefore$ Both u and v satisfies Laplace equation

  Hence u and v are Harmonic functions .

- Show that the function $\boldsymbol{u(x,y) = sinxcoshy}$ is harmonic .

  Find its harmonic conjugate $\boldsymbol{v(x,y)}$ and the analytic function $\boldsymbol{(z) = u + iv}$.

  Solution :

  Given, $\boldsymbol{u(x,y) = sinxcoshy}$

  $\boldsymbol{u_x = \cos x \cosh y}$ $\boldsymbol{u_y = \sin x \sinh y}$

  $\boldsymbol{u_{xx} = -\sin x \cosh y}$ $\boldsymbol{u_{yy} = \sin x \cosh y}$

  $\therefore \boldsymbol{u_{xx} + u_{yy} = 0}$

  Hence u is harmonic .

To find v(x,y):

_we know that $\boldsymbol{v = \int -u_y dx + \int u_x dy}$

$$\boldsymbol{v = \int -\left(\sin x \sinh y\right) dx + \int \left(\cos x \cosh y\right) dy}$$

$$\boldsymbol{= -\sinh y \int \sin x \, dx + 0}$$

$$\boldsymbol{= -\sin y\,(-cosx)}$$

$$\therefore \boldsymbol{v = \cos x \sinh y}$$

Now $\boldsymbol{f(z) = u + iv = sinxcoshy + i \cos x \sinh y}$

$$= \sin x \cos(iy) + \ i \cos x \left(\frac{\sin(iy)}{i}\right)$$

$$= \sin x \cos(iy) \ + \ cosx \sin(\text{iy})$$

$$= \sin(x + iy)$$

$$= sinz$$

$$\therefore f(z) = sinz \ .$$

CHAPTER 6

Real Life Applications of Analytic Function

6.1 Introduction A fundamental concept in complex analysis and analytical functions has extensive use in a variety of disciplines including mathematics, physics and more. These functions have derivatives of all orders within their domain and can be represented as power series. The analytical functions are useful in resolving practical problems and are essential for comprehending how functions behave in complicated contexts.

6.2 Applications of Analytic Functions:

There are various applications of analytic functions and some of them are added below:

In Complex Analysis:

Analytic functions are fundamental in complex analysis, where they play a central role in understanding complex numbers, mappings, and contour integrals. They provide tools for solving differential equations and studying the behaviour of complex systems.

In Electromagnetic Field Analysis:

In both physics and engineering electromagnetic fields are modeled and analyzed using the Analytic functions. Complex functions are frequently included in Maxwell's equations which describe the behavior of electric and magnetic fields. Through the representation of these domains as analytical functions, engineers can examine and refine designs for optical systems, microwave circuits and antennas.

In Signal Processing:

Analytical functions are used in signal processing to examine and work with the signals in the frequency and temporal domains. For instance, the Fourier transform is a widely used method in signal processing and uses complex exponentials which are analytical functions to break down a signal into its frequency components.

In Fluid Dynamics:

When studying fluid dynamics, analytical functions are very important especially when analyzing potential flows. Idealized models called potential flows are used to explain the motion of inviscid fluids like water and air. Fluid dynamics can solve complex flow problems like the flow around airfoils, ships and vehicles by expressing the velocity potential and stream function as the analytic functions.

In Financial Modeling:

Analytic functions are employed in financial modelling and quantitative finance to analyze and predict asset prices, risk factors and portfolio performance. Techniques such as the Black-Scholes model used for the pricing options and stochastic calculus used for the modelling of financial derivatives rely on the properties of the analytic functions to derive analytical solutions and make informed investment decisions.

In Computer Graphics:

Analytic functions are employed in computer graphics for rendering images, modeling surfaces, and creating visual effects. They are used to define curves and surfaces in three-dimensional space, enabling realistic rendering of objects and scenes.

In Control Systems:

In control theory, analytic functions are used to design and analyze feedback control systems for regulating the behaviour of the dynamical systems. By representing system transfer functions and controller designs as analytic functions, engineers can analyze the stability, performance and robustness properties of the control systems.

6.3 Conclusion:

Thus, one can conclude that, analytic functions serve as powerful mathematical tools with applications ranging from theoretical mathematics to practical engineering and computational sciences. Their properties and behavior provide valuable insights into complex systems and phenomena across diverse fields.

CHAPTER 7

Discussion and Conclusion

This dissertation has explored the theory and applications of analytic function and construction of analytic function using Milne Thomson Method, Direct Method and Exact Differential Equation Method within the framework of complex analysis. In mathematics, an analytic function, also known as a holomorphic function, is a complex function that is differentiable at each point in its domain. Essentially, it's a function that can be locally represented by a convergent power series. Key properties include being infinitely differentiable, continuous, and satisfying the Cauchy-Riemann equations.

A key outcome in the study of analytic functions is the realization that differentiability in the complex sense automatically guarantees infinite differentiability and the existence of power series representations. This is a direct consequence of satisfying the Cauchy-Riemann equations, which serve as the necessary and sufficient conditions for analyticity in a domain. These equations not only connect the real and imaginary parts of a function but also establish the deep interplay between differential calculus and the geometric behavior of functions in the complex plane.

In conclusion, the discussion of analytic functions underscores their critical role in complex analysis, demonstrating how their unique properties drive both mathematical theory and real-world applications.

_The concept of analytic functions stands as a cornerstone in the study of complex analysis. Throughout this dissertation, the exploration of analytic functions has revealed their profound significance in both theoretical mathematics and practical applications. An analytic function, defined as a function that is locally represented by a convergent power series and satisfies the Cauchy-Riemann equations within a given domain, exhibits unique and powerful properties unmatched in real analysis.

The research has emphasized how the differentiability of functions in the complex plane leads to remarkable outcomes, such as the existence of derivatives of all orders and the ability to express functions as power series. The fundamental theorems associated with analytic functions, including Cauchy's Integral Theorem, Cauchy's Integral Formula, Liouville's Theorem, and Morera's Theorem, provide deep insights into the structure and behavior of complex functions. These theorems not only enrich the mathematical theory but also have practical implications in various fields like fluid dynamics, electromagnetic theory, aerodynamics, and quantum mechanics.

Furthermore, the dissertation has demonstrated that analytic functions are essential in understanding conformal mappings, harmonic functions, and potential theory, which are widely applied in engineering and physical sciences. The intrinsic link between differentiability, integrability, and harmonicity in complex analysis showcases the elegance and depth of analytic functions.

In summary, the study of analytic functions in complex analysis forms a vital framework for advancing mathematical understanding and solving real-world problems. Their exceptional properties and wide-ranging applications continue to inspire further esearch and development in mathematics and its applied domains.

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