A STUDY ON APPLICATIONS OF NUMERICAL METHODS

Dissertation submitted to the Department of Mathematics in partial fulfillment of the requirements for the award of the degree of Master of Science in Mathematics

MAHAPURUSHA SRIMANTA SANKARADEVA VISHWAVIDYALAYA
NAGAON, ASSAM

SUBMITTED BY
ANKUR HIRA
Roll No.: MAT-07/23
M.Sc. 4th Semester
Session: 2023-2025

UNDER THE GUIDANCE OF

DR. MIRA DAS

ASSISTANT PROFESSOR

DEPARTMENT OF MATHEMATICS

MSSV,NAGAON

CONTENT

Certificate

This is to certify that **ANKUR HIRA** bearing **Roll No MAT-07/23** and **Regd. No. MSSV-0023-101-001542** has prepared his dissertation entitled “**APPLICATIONS OF NUMERICAL METHODS**” submitted to the Department of Mathematics, **MAHAPURUSHA SRIMANTA SANKARADEVA VISWAVIDYALAYA**, Nagaon, for fulfilment of MSc. degree, under guidance of me and neither the dissertation nor any part thereof has submitted to this or any other university for a research degree or diploma.

He/She fulfilled all the requirements prescribed by the department of Mathematics.

Supervisor
(DR. MIRA DAS)
Assistant Professor
Deepartment of Mathematics
MSSV, Nagaon (Assam)
E-mail: miradas622@gmail.com

DECLARATION

I, ANKUR HIRA , bearing ROLL No- MAT-07/23 student of final semester, Department of Mathematics , MAHAPURUSHA SRIMANTA SANKARADEVA VISHWAVIDYALAYA (MSSV), do hereby declare that the work incorporated in this dissertation entitled "**APPLICATIONS OF NUMERICAL METHODS**", for the award of degree of Master of Science in Mathematics, has been carried out and interpreted by me under the supervision of DR. MIRA DAS , Assistant Professor, Department of Mathematics , (MSSV) Nagaon. This dissertation is original and has not been submitted by me for the award of degree of diploma to any other University or Institute. I have faithfully and accurately cited all my sources, including books, journals, handouts and unpublished manuscripts, as well as any other media, such as the internet, letters or significant personal communication.

ANKUR HIRA
Roll No: MAT-07/23
M.Sc. 4th semester
Department of Mathematics, (MSSV)

ACKNOWLEDGEMENT

First and foremost, with pride and pleasure, I express my sincere and profound gratitude to my supervisor **Dr. MIRA DAS** ma'am, Assistant Professor (MSSV), for giving me the opportunity to work under his valuable guidance throughout the dissertation period.

I would like to acknowledge and thanks to the head of the department, Dr. RAJU BORDOLOI Sir for his valuable advice and encouragement during my study period. My sincere gratitude also goes to all the Faculty and staff of the Department of Mathematics, (MSSV) for sharing their ideas, thoughts, information throughout the completion of my dissertation period.

This acknowledgement extends to family, friends and colleagues whose support shaped this document. The acknowledgement for the dissertation file is recognition of the collaborative efforts that turned ideas into written words.

It's a token of gratitude to those who played a role, big or small, in bringing this document to completion.

ANKUR HIRA
Roll No: MAT-07/23
M.sc. 4th semester, Department of Mathematics,
MSSV, Nagaon (Assam)

ABSTRACT

Numerical methods play a important role in solving complex mathematical problems that lack analytical solutions. These techniques are essential in various scientific, engineering, economic, and biological domains, where exact solutions are either impossible or impractical to obtain. Numerical methods modified continuous problems into discrete ones, making them solvable using algorithms and digital computation. In engineering, these methods are employed for structural analysis, heat transfer, fluid dynamics, and stress analysis through finite difference and finite element methods. In physics, they aid in simulating wave propagation, quantum mechanics, and electromagnetic field behavior. In biology, models of population dynamics, epidemic spread, and enzyme kinetics are solved using techniques like Euler's method, Runge-Kutta methods, and numerical integration. Economics and finance rely heavily on numerical optimization, interpolation, and differential equation solvers to model market behavior, forecast trends, and manage risk. Numerical techniques such as Newton-Raphson and bisection methods are critical in root-finding problems arising in various disciplines, while numerical linear algebra supports data analysis, computer graphics, and machine learning.

Overall, numerical methods bridge the gap between theoretical models and practical application by offering effective and approximate solutions to real-world problems. Their flexibility and adaptability to different fields underscore their importance in modern computational science and engineering.

In chapter I, In this chapter I have covered the introduction and basic definition, importance and relevance and also difference between analytical and numerical approaches of application of numerical methods.

In chapter II, Here we have discussed types of errors, error propagation and significance in real-word problems of error analysis.

In chapter III, In this chapter we discussed likely covers root finding applications practical examples in engineering and science and algorithms like bisection, Newton-Raphson etc .

In chapter IV, In this chapter we discussed about the methods of solution of first order ordinary differential equations and its application in biology, economics, mechanics. Also we discussed Picard method, Eulers method, Eulers modified method, Runge-Kutta, Thomson method .Here we gave some real life modeling examples.

In chapter V, Here we have discussed methods of solution of partial differential equations and heat transfer, wave propagation and stress analysis. Also we discussed finite difference and finite element applications.

REVIEW OF LITERATURE :

Burden and Fairs(2011).

The evolution of numerical methods can be traced back to early efforts by Newton, Euler, and Gauss, who introduced initial forms of interpolation, numerical integration, and differential solvers. With the advent of computers in the 20th century, the field expanded rapidly. According to Burden and Faires (2011), the need for computational approaches grew with the increasing complexity of models, particularly in fluid dynamics, thermodynamics, and structural mechanics**.[1]**

Press et al. (2007).

Press et al. (2007), in Numerical Recipes, provided one of the most influential references, consolidating algorithms such as the Runge-Kutta family, Newton-Raphson, Gauss-Seidel iteration, and numerical integration techniques like Simpson's and Trapezoidal rules. These algorithms became the building blocks for scientific computing.**[2]**

Zienkiewic and Tylor (2005).

Numerical methods, especially the Finite Element Method (FEM), have transformed the field of structural engineering. FEM discretizes continuous domains into smaller elements, enabling the simulation of stresses, strains, and deformations in structures. Zienkiewicz and Taylor (2005) extensively documented FEM's use in analyzing bridges, buildings, and mechanical components. For dynamic simulations involving time-dependent processes, implicit and explicit time integration schemes are widely used.**[3]**

Incropera et al. (2007).

Finite Difference Method (FDM) and Finite Volume Method (FVM) are widely used in solving heat conduction and fluid flow problems. Incropera et al. (2007) demonstrated FDM's effectiveness in solving transient and steady-state heat conduction equations.**[4]**

Versteeg & Malalasekera, 2007.

In fluid mechanics, Computational Fluid Dynamics (CFD) employs FVM to solve the Navier-Stokes equations, enabling simulation of aerodynamic flows, combustion, and turbulence**.[5]**

Taflove & Hagness, 2005

In physics, numerical solutions are indispensable in areas where analytical models are intractable. For example, quantum mechanics involves solving the Schrödinger equation for complex systems, often requiring the use of Crank-Nicolson or finite difference time domain (FDTD) methods. Maxwell's equations, central to electromagnetics, are solved using FDTD or FEM in antenna design and wave propagation studies **.[6]**

Muray (2002)

Murray (2002) discussed the application of numerical solvers in population dynamics and disease modeling. The SIR (Susceptible-Infectious-Recovered) model, for example, is frequently solved using explicit Euler or fourth-order Runge-Kutta methods to predict epidemic trends. Sensitivity analysis and parameter estimation are also conducted through numerical optimization**.[7]**

Fernández et al.,(2009)

FEM is widely used in medical image reconstruction (e.g., CT or MRI scan processing) and biomechanical modeling, such as simulating blood flow in arteries or stresses in bones and tissues. These models are used for diagnostics, surgical planning, and device design**.[8]**

Judd (1998)

Judd (1998) emphasized the need for numerical dynamic programming and shooting methods to solve models in economic growth and policy analysis. These models are typically described by systems of differential or difference equations.**[9]**

Wilmott et al. (1995)

In finance, the valuation of options and derivatives often involves solving the Black-Scholes PDE. Due to the lack of closed-form solutions in most real-world cases, finite difference methods are employed to obtain numerical prices. Wilmott et al. (1995) developed several stable and accurate schemes for this purpose**.[10]**

Anderson & Woessner, (1992)

Environmental modeling includes the simulation of groundwater flow, weather prediction, and climate modeling. Numerical weather prediction models rely on solving coupled PDEs for atmospheric variables using FDM or spectral methods. Groundwater and pollutant transport models use FEM and FVM to simulate subsurface flow and contamination spread **.[11]**

Introduction

Numerical methods are important and essential mathematical tools that allow us to find approximate solutions to complex mathematical problems which cannot be solved analytically. In many real-world scenarios, like- engineering simulations, climate modeling, and economic forecasting, the governing equations are too complicated for traditional analytical techniques. Numerical methods give a practical and reliable way to analyze such problems by converting them into computable algorithms.

The significance of numerical methods has increased rapidly with the advancement of digital computing. While classical mathematics targeted on exact formulas, numerical techniques embrace approximation to tackle real-world challenges with acceptable accuracy. These methods are particularly important when dealing with non-linear equations, large data sets, partial differential equations, and multi-variable systems that arise frequently in science and engineering.

In engineering, numerical methods are broadly used to simulate structural behavior, analyze heat and fluid flow, and design electrical systems. For example, the finite element method (FEM) is a powerful tool used in civil and mechanical engineering for stress analysis and vibration studies. Similarly, the finite difference method (FDM) and finite volume method (FVM) are widely applied in thermal simulations and fluid dynamics.

In the physical and natural sciences, numerical techniques allow researchers to model phenomena that are either too complex or confused for analytical solutions. From modeling planetary motion in astrophysics to predicting wave propagation in materials, numerical methods enable scientists to gain insight into dynamic systems over time.

The field of biology also gives benefits greatly from numerical approaches, especially in population dynamics, the spread of infectious diseases, and enzyme kinetics. These biological systems are often reported by systems of ordinary or partial differential equations that are best handled using numerical solvers like the Runge-Kutta methods or Euler's method.

Economics and finance depend heavily on numerical methods for tasks like as risk modeling, optimization, and option pricing. Financial derivatives, for example, are valued using iterative techniques like Newton-Raphson or numerical integration methods when closed-form solutions are unavailable. Linear programming and numerical optimization techniques helps us to solve resource allocation and cost minimization problems.

At the core of numerical methods lies the principle of discretization, which involves breaking continuous problems into finite parts. These parts are then solved using computational algorithms. This process, although approximate, can yield solutions with high accuracy depending on the chosen method and computational resources.

Another major advantage of numerical methods is their flexibility. Modern software tools such as MATLAB, Python (NumPy and SciPy), Mathematica, and C++ libraries have made the execution of numerical algorithms more reachable and efficient. These tools allow researchers and practitioners to imitate complex systems, visualize solutions, and conduct sensitivity analyses in ways that were not possible with manual calculations.

Despite their many benefits, numerical methods also present certain challenges. These include issues related to numerical stability, error propagation, convergence, and computational cost. Improper application or poor choice of method parameters can lead to inaccurate or misleading results. Therefore, a sound understanding of the underlying mathematical principles and error behavior is essential for essential use.

These methods bridge the gap between theoretical mathematics and practical problem-solving in the modern world. They are not only tools of approximation but are also crucial components in decision-making, innovation, and the advancement of science and technology. Their widespread application across disciplines demonstrates their versatility and enduring importance in both academic research and industrial practice.

CHAPTER-1

PRELIMINARIES

1.1 INTRODUCTION

In the fields of science, engineering, economics, and applied mathematics, numerous problems are represented using mathematical equations. Unfortunately, obtaining precise analytical solutions for these equations is frequently unattainable due to their intricate nature or the specific characteristics of the data being analyzed. Numerical methods provide a means to find approximate yet practical solutions using computational algorithms. Numerical methods employ iterative procedures to approximate numerical solutions for mathematical problems that cannot be solved analytically. These techniques are essential in practical problem-solving, particularly in fields that heavily rely on modeling and simulations. This article provides comprehensive analyses of numerical methods, covering their various types, applications, benefits, and obstacles.

Numerical methods have become the fundamental tools for solving complex scientific and engineering problems in the modern era. In a world increasingly reliant on computation, these methods offer a practical, flexible, and scalable approach to handling complex mathematical problems that resist analytical solutions, whether it's predicting the trajectory of a space craft, designing a bridge, iimproving a supply chain or analyzing climate dada, numerical methods play a crucial role in providing reliable, approximate solutions to real-world-problems.

In contrast to traditional mathematics, which focuses on precise solutions, numerous real-world systems are too intricate, chaotic, or data-heavy to be solved exactly. Instead, engineers and scientists-turned-computational experts rely on step-by-step numerical algorithms to obtain accurate solutions within acceptable error margins.

However the key characteristics of Numerical methods are-

(i). They give approximate solutions rather than exact ones.

(ii). They are implemented using algorithms, that are suitable for computers.

(iii). They involve iteration and Convergence towards the desired result.

(iv) They aim to balance accuracy stability and computational efficiency.

This data presents an extensive examination of how numerical methods are applied across various fields and problem types. I explore both the theoretical foundations and practical implementations of these techniques supported by real-world examples, industry practices, and informative case studies. The goal is not only to understand how these methods work but also to appreciate their value in transforming abstract problems into tangible solutions.

1.2 IMPORTANCE AND RELEVANCE

Numerical methods form a fundamental idea about scientific computing, offering practical tools for solving mathematical problems that are difficult or impossible to examine analytically. In a world increasingly driven by data and complex systems, numerical methods give essential algorithms for estimating solutions to equations, simulating physical systems, and iimproving performance across a broad spectrum of disciplines.

(i)Necessity for Solving Real-World Problems

Many real-world problems result in mathematical models that cannot be solved using analytical methods. For example, differential equations modeling weather systems, fluid dynamics, or financial markets rarely yield closed-form solutions. In such cases, numerical methods such as finite difference methods, Runge-Kutta schemes, or numerical integration are employed to approximate solutions with sufficient accuracy.

In engineering disciplines, stress-strain analysis in complex geometries, heat conduction problems, and dynamic systems modeling often rely heavily on numerical approaches such as the Finite Element Method (FEM). These methods help engineers to predict behavior, test designs virtually, and minimize costly physical experimentation.

(ii) Bridging Theory and Practice

While mathematical theory gives a conceptual framework, numerical methods give practical implementation. For example, although the theory of linear algebra offers solutions to systems of equations, numerical methods like LU decomposition such as the Gauss-Seidel method permit us to implement these solutions in computational environments. This synergy assure that theoretical models have real-world applicability, especially in scenarios involving large datasets or high-dimensional systems.

(iii) Computational Efficiency and Accuracy

Modern numerical algorithms are designed for efficiency and precision, particularly with the aid of high-performance computing. Adaptive methods dynamically adjust step sizes resolutions to improve accuracy while reducing computational cost. For example, adaptive quadrature upgrades numerical integration over irregular domains, while adaptive mesh refinement in FEM enhances accuracy in regions requiring finer resolution.

These methods are also critical in controlling and quantifying errors. Understanding truncation, round-off, and propagation errors is vital in clearify the reliability of results obtained through numerical simulations, especially in critical applications like structural safety or climate prediction.

(iv) Applicability Across Disciplines

The relevance of numerical methods spans across almost all domains of science and engineering:

- Physics and Engineering: Solving Maxwell’s equations, heat transfer and fluid mechanics.
- Finance and Economics: Option pricing using the Black-Scholes equation and portfolio optimization.
- Biology and Medicine: Modeling biological systems, neural networks, and drug diffusion processes.
- Data Science and AI: Optimization algorithms, numerical linear algebra for machine learning models and also statistical approximations.

Such diversity in applications makes numerical methods is a universal toolset for researchers and professionals alike.

(v) Enabling Innovation Through Simulation

Numerical simulations permit researchers to scan hypotheses, predict system behavior under different scenarios, and test new designs or policies before physical implementation. This capability is especially valuable in fields where experimentation is expensive, dangerous and impractical—such as aerospace design, nuclear reactor modeling, or pandemic forecasting.

However, the rise of digital twins—virtual replicas of physical systems—leans heavily on numerical methods to simulate real-time behavior and inform decision-making processes.

1.3 DIFFERENCE BETWEEN ANALYTICAL AND NUMERICAL APPROACHES

In the study and application of mathematical modeling and problem-solving, two fundamental approaches exist, analytical methods and numerical methods. Both define the purpose of finding solutions to mathematical problems, but they vary significantly in their methodology, scope, accuracy, and applicability. Understanding these differences is important for selecting the appropriate technique in scientific and engineering applications.

(i) Definition and Nature of Solutions

Analytical methods which also called exact methods aim to find closed-form expressions for mathematical problems. These solutions are expressed in terms of known functions and constants, providing exact results. For example, solving a quadratic equation analytically yields an exact solution using the quadratic formula.

In simple word, numerical methods include approximating solutions using algorithms and computational procedures. These techniques do not target for exact symbolic expressions but instead give numerical values that approximate the true solution to a desired level of accuracy. For example, solving a nonlinear equation using the Newton-Raphson method involves iterating to converge on a root.

(ii) Applicability to Real-World Problems

Analytical solutions are often viable only for simplified or idealized problems. For example, the heat equation in one dimension with simple boundary conditions can be solved analytically using separation of variables. However, when extended to more realistic scenarios—like as irregular geometries, variable material properties, or nonlinearities—analytical methods usually become impractical, difficult or impossible.

Numerical approaches excel in handling such complex, real-world problems. They gives for the solution of equations and systems that have no closed-form solution, including nonlinear differential equations, systems with complicated boundaries, and problems involving high-dimensional data. As a result, numerical methods are broadly used in computational fluid dynamics, structural analysis, weather prediction, and other fields requiring practical solutions.

(iii) Computational Resources and Implementation

Analytical solutions basically require symbolic manipulation and performed manually or using algebraic software like Mathematica or Maple. Once derived, these solutions are effective to evaluate and give insight into the general behavior of a system.

On the other hand, numerical methods need computational power and algorithm design. These methods involve discretization for e.g., time-stepping or meshing, iteration, and convergence checks. Software tools like as MATLAB, Python (NumPy/SciPy), and C++ are frequently used for implementing numerical algorithms. While they demand more computational resources, they offer the flexibility to model more realistic systems.

(iv) Accuracy and Error Considerations

Analytical solutions, when available, are exact and do not suffer from numerical errors. However, they are limited and counted by the assumptions required to make a problem solvable.

Numerical methods are basically approximate and subject to various sources of error, including:

- Truncation error from approximating continuous problems using discrete steps.
- Round-off error due to limited precision in computer arithmetic.
- Stability and convergence issues if the algorithm is not carefully chosen.

Despite these limitations, this methods can be made highly accurate and perfect by refining discretization, improving algorithms, and using error estimation techniques.

(v) Insight vs. Practicality

Analytical solutions often give theoretical insight into how parameters impact the system, helping researchers acknowledge underlying mechanisms. For example, an exact formula for the displacement in a vibrating string can reveal how wave speed depends on tension and mass density.

Numerical methods are realistic tools, enabling engineers and scientists to obtain operable answers when analytical approaches fail. While they may lack the elegance of closed-form expressions, they give required capabilities for design, simulation, and prediction in the real world.

CHAPTER 2

ERROR ANALYSIS

2.1 INTRODUCTION

Error analysis is a vital component of numerical methods, that helps in understanding the accuracy and reliability of numerical computations. In real-world problems, exact solutions are often impossible and difficult to obtain analytically due to complex mathematical models or lack of precise data. It give approximate solutions, and error analysis helps quantify the deviation between the numerical result and the true and exact solution.

2.2 TYPES OF ERRORS

Numerical methods concern the use of approximations to solve mathematical problems. These approximations introduce certain types of errors. Understanding these errors is critical for improving accuracy and reliability in computations. The types of errors in numerical methods include:

(i) Round-off Error

Definition: This error occurs due to the limitation of storing numbers in computers with finite precision. Computers cannot represent most real numbers exactly, especially repeating decimals.

Example: Taking the number π as 3.14159 instead of its infinite decimal expansion introduces round-off error.

Impact: Small in individual calculations but can accumulate and become significant in large computations or iterative processes.

(ii) Truncation Error

Definition: Truncation error appears when an infinite process is approximated by a finite one. It is the difference between the exact mathematical value and the approximate value used in the numerical method.

Types: Local Truncation Error: Error in a single step of a numerical process.

Global Truncation Error: Cumulative effect of local truncation errors over many steps.

Example: Approximating a derivative using finite differences:

$$f'(x) \approx \frac{f(x + h) - f(x)}{h}$$

(iii) Discretization Error

Definition: This error results from replacing a continuous model for e.g., a differential equation with a discrete one. It is closely related to truncation error but refers more widely to the errors from the numerical representation of continuous systems.

Example: Solving differential equations using numerical grids introduces discretization error based on grid size.

Impact: Can be reduced by refining the discretization (e.g. using smaller step sizes or more points).

(iv) Algorithmic Error

Definition: This error comes from using inappropriate algorithms. Even if input data is accurate, poor algorithm design may lead to large inaccuracies.

Example: Using an unstable method to solve a system of linear equations may expand small errors.

Prevention: Choose algorithms with good stability and convergence properties.

(v) Modeling Error

Definition: Occurs when the mathematical model used does not accurately represent the real-world system. This is not directly a computational error but affects the accuracy of numerical solutions.

Example: Using linear equations to model a nonlinear phenomenon.

Minimization: Develop more realistic, appropriate mathematical models for the situation.

(vi) Human Error

Definition: Errors due to mistakes in coding, data entry and interpretation of results.

Example: Entering incorrect values, choosing wrong step size and misapplying numerical formulas.

Solution: Careful validation, testing, and debugging.

2.2 ERROR PROPAGATION

Error propagation relates to the way initial or intermediate errors in numerical calculations influence the final result. When multiple operations are performed, even a small error in input or an intermediate step can cause larger deviations in the output. This is very important in iterative algorithms and complex computations.

2.2.1. Sources of Error Propagation

(i) Initial Input Errors

Errors in starting values such as approximations or measurements carry through all subsequent steps.

(ii) Round-off Errors

Since computers have finite precision, numbers are rounded during each operation. These small round-off errors accumulate and propagate.

(iii) Truncation Errors

When infinite processes like integrals and derivatives are approximated using finite steps, the truncation error from one step influences the next.

(iv) Computational Algorithms

Some algorithms are numerically unstable, meaning they expand small errors as the computation proceeds.

2.2.2. Examples of Error Propagation

(i) Iterative Methods

In the methods like Newton-Raphson or Gauss-Seidel, error from one iteration affects the next. If it is not controlled, this result may in divergence or slow convergence.

(ii) Solving Linear Systems

When solving, small errors in matrix and vector can cause large errors in solution, especially if it is ill-conditioned.

(iii) Numerical Differentiation

Approximating derivatives involves subtracting nearly equal numbers, which can significantly amplify round-off errors.

2.2.3. Analysis of Error Propagation

(i) Forward Error Analysis

Estimates how much the final result deviates from the true value due to initial errors.

(ii) Backward Error Analysis

Determines what exact input would have produced the observed (approximate) output, helping analyze stability.

(iii) Condition Number

For linear systems and functions, the condition number measures sensitivity to input changes where high condition numbers indicate greater error propagation.

2.2.4. Controlling Error Propagation

Use stable algorithms: Prefer numerically stable methods for e.g., LU decomposition over Gaussian elimination in some cases.

Reduce step sizes: In methods like Euler's or Runge-Kutta, smaller steps reduce truncation errors.

Use higher precision: Floating-point precision can be increased when necessary.

Error estimation: Apply techniques to estimate and correct accumulated errors.

2.2.5. Real-World Importance

In real-world applications like as engineering simulations, climate modeling and space navigation, even tiny propagated errors can cause system failures. Hence, understanding and mitigating error propagation is vital.

2.3 SIGNIFICANCE IN REAL-WORLD PROBLEMS

Numerical methods are necessary tools for solving complex mathematical problems that cannot be addressed analytically. In real-world scenarios—where exact solutions are often impossible, difficult and impractical—numerical methods give efficient, accurate, and implementable solutions across science, engineering, finance etc.

(i) Solving Real-World Equations

Many physical, economic, and biological systems are modeled using equations for e.g., nonlinear, differential and algebraic. Most of these equations cannot be solved exactly using analytical techniques.

Example: Solving non-linear equations in fluid dynamics or heat transfer.

Simulating the spread of diseases using differential equations.

Numerical methods, like the Newton-Raphson method or finite difference methods, make it possible and easier to find approximate solutions where exact ones are infeasible.

(ii) Engineering Applications

In civil, mechanical, electrical, and aerospace engineering, numerical methods are widely used in simulations and design.

Examples: Finite Element Method (FEM): Used in structural analysis to determine stress and strain in bridges, aircraft and buildings.

Computational Fluid Dynamics (CFD): Used in designing aerodynamic vehicles by simulating air flow.

Signal Processing: Algorithms for filtering and analyzing signals depend on numerical techniques.

These applications do save time and cost by permiting virtual testing and optimization before physical production.

(iii) Weather Forecasting and Climate Modeling

Meteorological predictions depend on solving complex systems of partial differential equations that describe atmospheric behavior. These equations are solved numerically using supercomputers.

Significance: Helps in predicting storms, rainfall, and temperature changes.

Critical for disaster preparedness and agriculture planning.

(iv) Medical and Biological Sciences

Numerical methods are essential in medical imaging, diagnostics, and biological modeling.

Examples: Image reconstruction in CT scans and MRIs.

Pharmacokinetics models to simulate how drugs move through the human body.

Population dynamics models in ecology and epidemiology.

These applications improve diagnosis, treatment planning, and healthcare outcomes.

(v) Finance and Economics

Numerical techniques are used to model market behavior, assess risks, and price complex financial instruments.

Examples: Monte Carlo simulations in risk assessment.

Numerical integration in option pricing models (e.g., Black-Scholes).

Optimization algorithms in portfolio management.

They enable better financial decision-making and forecasting.

(vi) Space and Aerospace Industry

In space missions, exact calculations for trajectory, fuel efficiency, and communication delay are impractical and difficult. Numerical methods permit for precise simulations and adjustments.

Example: NASA uses numerical methods to calculate spacecraft trajectories and to model re-entry conditions.

(vii) Industrial Automation and Robotics

Real-time control systems, robotic movement, and artificial intelligence depend on numerical methods for solving systems of equations, path planning, and sensor data processing.

(viii) Everyday Technologies

From GPS navigation to video game physics engines, numerical algorithms run in the background of many modern technologies we use daily.

CHAPTER 3

ROOT FINDING APPLICATIONS

3.1 Introduction

Root finding applications includes determining the values of variables that satisfy equations of the form f(x)=0, where analytical solutions are critical. These methods are critical in fields like engineering, physics, and economics for solving real-world nonlinear problems.

3.2 PRACTICAL EXAMPLES IN ENGINEERING AND SCIENCE

In numerical methods, root-finding techniques are used to determine the values of variables (roots) that satisfy equations of the form:

$$\mathbf{f(x) = 0}$$

Many real-world problems like in engineering and science lead to such equations. However, most of these equations do not have exact analytical solutions and require numerical methods like the Bisection Method, Newton-Raphson Method, or Secant Method for approximate solutions.

1. Engineering Applications

(i) Electrical Engineering – Circuit Analysis

In non-linear electrical circuits involving diodes or transistors, the current-voltage relationship is governed by non-linear equations.

Example: The Shockley diode equation:

$$I = I_S(e^{\frac{V}{n}V_t} -1)$$

(ii) Mechanical Engineering – Stress and Deflection

Design problems involve finding the point where deformation and stress meets a safety threshold, often resulting in nonlinear equations.

Example: Solving for the critical load in a buckling column using characteristic **equations:**

$$\tan(\lambda L) = \frac{\lambda L}{F}$$

(iii) Civil Engineering – Beam and Foundation Design

In structural design, deflection of beams or soil settlements often leads to transcendental equations needing numerical solutions.

Example: Solving for settlement depth using soil mechanics equations that are nonlinear.

2. Scientific Applications

(i) Physics – Motion and Kinetics

Equations in classical mechanics need roots to determine time, velocity, or temperature.

Example: Finding the time when a projectile hits the ground:

$$s(t) = ut + \frac{1}{2}at^2$$

(ii) Chemistry – Equilibrium Calculations

Chemical equilibrium problems involve solving nonlinear equations for concentration values.

Example: Solving the equation for pH in a weak acid solution:

$$[H^+] = \sqrt{K_a.C}$$

(iii) Environmental Science – Root Zone Modeling

Equations modeling water flow through soil layers or pollutant diffusion often require root-finding to calculate depth or time of occurrence.

3.3 Computer Simulations and Control Systems

In simulations of dynamic systems, root-finding is used to determine points of system stability or control transitions.

Example: In control systems, finding when the system characteristic equation equals zero:

$$\det(sI - A) = 0$$

3.4 ALGORITHMS (BISECTION, NEWTON-RAMPHSON etc)

Common Root-Finding Algorithms

(i) Bisection Method

Based on the **Intermediate Value Theorem:** if f(a).f(b) < 0, there's at least one root in [a, b].

The interval is repeatedly halved to narrow in on the root.

Formula:

Midpoint: m = (a + b)/2

Update interval: If f(a)·f(m) < 0, set b = m;

Else, set a = m

Advantages:

Simple and always converges (for continuous functions)

No need for derivatives

Disadvantages:

Slow (linear convergence),

Needs a bracketing interval,

Used when function behavior is unknown but continuity is assured.

(ii) Newton-Raphson Method

Uses the first-order Taylor expansion to approximate the root.

Requires the function's derivative.

Formula:

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

Advantages:

Very fast (quadratic convergence near root)

Requires only one initial guess

Disadvantages:

Needs $f'(x)$, which may be complex to compute

May fail if $f'(x) \approx 0$ or if starting guess is poor

Widely used in engineering simulations and control systems.

3.3 Secant Method

An approximation to Newton-Raphson which avoids derivative computation.

Uses previous two estimates to form a secant line.

Formula: $x_{n+1} = x_n - f(x_n)(x_n - x_{n-1}) / [f(x_n) - f(x_{n-1})]$

Advantages:

Faster than bisection

Doesn’t require derivatives

Disadvantages:

Slower than Newton- Raphson,

May diverge or oscillate,

Preferred when derivatives are difficult or impossible to calculate.

Comparison Table:-

METHODS	CONVERGENCE REQUIRES	DERIVATIVE	GUARANTEED CONVERGENCE	SPEED
BISECTION	Linear	No	Yes	Slow
NEWTON-RAPHSON	Quadratic	Yes	No	Fast
SECANT	Super Linear	No	No	Moderate

CHAPTER-4

METHODS OF SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

4.1 Introduction

First-order ordinary differential equations (ODEs) roll out naturally in a wide range of scientific and engineering disciplines. These are mathematical equations that connect with a function with its first derivative. These equations are critical in modeling various real-life dynamic systems where the rate of change of a quantity is of interest. Numerical methods give important tools to solve such equations, especially when analytical solutions are difficult to obtain. A first-order ordinary differential equation (ODE) is an equation involving a function and its first derivative. It takes the general form:

$$\frac{dy}{dx} = f(x, y)$$

These equations describe how a quantity changes w.r.t. another and are used in science, economics, and engineering.

4.2 Applications in biology, economics, mechanics :-

(i) In Biology:-

Population Growth Models: The logistic and exponential growth equations give how populations grow over time.

Example: $$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

Spread of Diseases (Epidemiology): Models like SIR used first-order ODEs to track susceptible, infected, and recovered individuals over time.

(ii)In Economics:-

Compound Interest and Investment Models: Used to predict future value of investments.

Example: $$\frac{dA}{dt} = rA$$

Market Equilibrium Dynamics: Price adjustment models are used in first-order ODEs to describe how prices change over time based on supply and demand imbalances.

(iii)Mechanics:-

Velocity and Acceleration: Newton's second law conduct to first-order ODEs when force is a function of velocity or position.

Example: $$m\frac{dV}{dt} = -kv$$

Cooling and Heating (Newton's Law of Cooling):

$$\frac{dT}{dt} = -k(T - T_{ambient})$$

4.2 PICARD METHOD, EULERS METHOD, EULERS MODIFIED METHOD, RUNGE-KUTTA, MILNE THOMSON METHOD

(i) PICARD METHOD

The Picard method, also known as the method of successive approximations, is a technique for approximating solutions to first-order ordinary differential equations with initial conditions. In short, It is an iterative technique used to find approximate solutions of first-order ordinary differential equations (ODEs). It is especially convenient when an exact analytical solution is difficult to obtain. This method is based on successive approximations and relies on the Picard–Lindelöf theorem, which gives the existence and uniqueness of solutions under certain conditions.

Standard Form :-

Picard's method applies to differential equations written in the form:

$$\frac{dy}{dx} = f(x,y), \quad y(x_0)=y_0$$

Methodology

Initial Approximation:

$$y_0(x) = y_0$$

$$y_0(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))dt$$

Each iteration gives an accurate approximation of the true solution.

Example

Consider the equation:

$$\frac{dy}{dx} = x + y,\ y(0) = 1$$

First Approximation:

$$y_0\ (x) = 1$$

$$y_0\ (x) = 1 + \int_0^x (t + 1)dt = 1 + \left[\frac{t^2}{2} + t\right]_0^x = 1 + \frac{x^2}{2} + x$$

Use $y_1(t)$ in the next integral to find $y_2(x)$,

Graphical Interpretation

Picard's method builds a curve that gets closer to the actual solution by integrating using the previous approximation. Each step gives the curve.

Applications

Numerical Analysis: Solving nonlinear or unsolvable ODEs analytically.

Biology: Modeling population growth or disease spread.

Economics: Modeling dynamic systems like interest rates, market trends.

Engineering: Control systems, heat transfer, and also electrical circuits.

Advantages:

Simple to implement.

Clear algorithmic structure.

Convenient for providing solutions exist and these are unique.

Guarantees convergence under suitable conditions.

Limitations:

Requires the function to be continuous and satisfy the Lipschitz condition.

Convergence may be slow.

Not efficient for real time or large scale computation.

Not practical for stiff differential equations.

Therefore this method is a foundational technique in the numerical solution of ODEs. Though not very used for practical computation due to convergence issues, it plays an important role in the theoretical understanding of existence and uniqueness of differential equations and also serves as the basis for more advanced methods.

(ii) EULERS METHOD

One of the simplest and easy fundamental numerical methods is Euler’s Method, named after the Swiss mathematician Leonhard Euler. Euler's method is a numerical technique used to approximate solutions to first-order ordinary differential equations (ODEs). It's a simple and instinctive method that uses a step-by-step approach to estimate the solution of an ODE, when analytical solutions are difficult and impossible to find. The method relies on the method of linear approximation, using the derivative at a known point to estimate the function's value at a nearby point.

Concept and Formula

Euler's method approximates the solution of an ODE by stepping from the initial point (x_0, y_0) forward in small increments of size h (step size).

The general formula is:

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$x_{n+1} = x[n + h\ ]$$

Where

y_n is the approximate solution at x_n,

$f(x_n, y_n)$ is the slope at that point,

h is the chosen step size.

The smaller the value of h is the more accurate the result, at the cost of more computations.

Step-by-Step Example

Problem:

$$\frac{dy}{dx} = x + y,\ y(0) = 1,\ h = 0.1$$

We will compute the values at x=0, 0.1, 0.2, 0.3.

Step 0: $x_0 = 0,\ y_0 = 1$

Step 1: $y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$

Step 2: $y_2 = 1.1 + 0.1(0.1 + 1.1) = 1.1 + 0.1(1.2) = 1.22$

Step 3: $y_3 = 1.22 + 0.1(0.2 + 1.22) = 1.22 + 0.1(1.42) = 1.362$

Hence, the approximate solution at $x=0.3$ is $y(0.3) \approx 1.362$.

Graphical Interpretation:

Euler’s method go after the slope of the tangent line at each step to move forward. The method “walks along” the curve, estimating each next point from the current one. Smaller step sizes give the true curve more closely, reducing error.

Accuracy and Limitations:

Euler’s method is a first-order method, meaning the global error is proportional to the step size h. Then:

Smaller step size = better accuracy.

But too small increases computational time.

Limitations:

Low accuracy for large step sizes.

Can diverge for stiff (unstable systems).

Not suitable for highly sensitive systems without modification.

Requires small h for reliable results,increasing computational load.

Advantages:

Simple to understand and implement,

Computationally inexpensive for small problems,

Convenient as a base for more advanced techniques.

Applications:

Euler's method is used in:

Physics: Motion and force simulations,

Biology: Population models,

Finance: Growth/decay modeling,

Engineering: Circuit and control systems.

It is valuable in educational contexts to introduce numerical ODE solving and develop understanding before using more advanced methods.

Therefore it is a fundamental numerical technique for approximating solutions to first-order ODEs. While it may not offer high accuracy or efficiency, it gives as a crucial learning tool and a stepping stone toward more upgraded numerical methods. Understanding Euler's method gives a strong foundation in the numerical solution of differential equations and also in the practical application of mathematics to real-world problems.

(iii) EULERS MODIFIED METHOD

Euler's modified method, which is also known as the improved Euler's method or Heun's method, is a numerical technique used to approximate the solution of first-order ordinary differential equations (ODEs) It made upon the basic Euler’s Method by using an average of slopes to estimate the next value more accurately. It is basically used in engineering, physics, and computational sciences where analytical solutions are not possible, and a reliable approximation is required.

Initial Value Problem Form:

We consider a first-order Ordinary Differential Equations:

$$\frac{dy}{dx}= f(x, y),\ y(x_0)= y_0$$

Modified Euler’s Method Formula:

The method consists of two steps:

Step 1: Predictor (Euler’s Estimate)

$$y_{n+1}^{(p)}= y_n+ h\ f(x_n,y_n)$$

Step 2: Corrector (Average Slope)

$$y_{n+1}= y_n+ \frac{h}{2}[f(x_n,y_n) + f\left(x_{n+1},y_{n+1}^{(p)}\right)]$$

This method upgrades accuracy by averaging the slope at the beginning and end of the interval.

STEP 3: Step-by-Step Example

Given:

$$\frac{dy}{dx} = x + y, \quad y(0) = 1, \quad h = 0.1$$

We will compute y(0.1).

Step 1: Predictor

$$y^{(p)} = 1 + 0.1\,.(0 + 1) = 1.1$$

Step 2: Corrector

$$y_1 = 1 + \frac{0.1}{2}[(0 + 1) + (0.1 + 1.1)] = 1 + 0.05(1 + 1.2) = 1 + 0.05(2.2) = 1.11$$

So, $y(0.1) \approx 1.11$

Step 5: Accuracy and Error

Order of Accuracy: 2nd order (better than basic Euler)

Local Truncation Error: $O(h^3)$

Global Error: $O(h^2)$

It is more accurate and stable than the Euler's Method, especially for nonlinear or rapidly changing functions.

Applications:

Physics: Motion of particles, heat conduction,

Biology: Population growth models,

Economics: Predictive models,

Engineering: Circuit analysis, fluid dynamics.

Advantages and Limitations:

Advantages:

More accuracy than Euler's method with the same step size,

Easy to implement,

Convenient for small to medium computational problems,

More stable and also reliable for moderate step sizes.

Limitations:

Requires two function evaluations per step which slower than basic Euler

Still less accurate than Runge-Kutta methods for stiff or highly nonlinear equations,

Not suitable for systems where high precision is needed over long intervals.

Therefore Modified Euler's Method gives a valuable improvement over the Euler's method. By using a predictor-corrector approach and averaging the slopes, it gives better results with only a small increase in computational effort. It gives a balance between simplicity and accuracy, making it a widely used method in numerical analysis for solving first-order ODEs. Understanding Modified Euler's Method helps to build the foundation for advanced numerical techniques and contributes to more reliable modeling in science and engineering.

(iv) RUNGE-KUTTA METHOD

The Runge-Kutta methods are a family of iterative numerical techniques used to approximate solutions to ordinary differential equations (ODEs). Specifically, they are used to find approximate values of a function at different points, given its initial value and a differential equation that defines its rate of change. The Runge-Kutta methods are valuable because it give a balance between accuracy and computational cost, making them a popular choice for solving a wide range of ODE problems. Many scientific and engineering problems involve solving first-order ordinary differential equations (ODEs) of the form:

$$\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$$

1) **The RK4 Formula**

The RK4 method estimates the value of y at the next point $x_{n+1} = x_n + h$ using:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Where:

$$k_1 = h.f(x_n, y_n)$$

$$k_2 = h.f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$

$$k_3 = h.f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$

$k_4 = h.f(x_n + h, y_n + k_3)$

h: step size

k_1, k_2, k_3, k_4 :estimates of the slope at different points within the interval

The final result is a weighted average of these slopes

2 Step-by-Step Example

Problem: $\frac{dy}{dx} = x + y$, y(0)= 1, h = 0.1

Step 1:

$k_1 = 0.1 \,.\, (0 + 1) = 0.1$

$k_2 = 0.1 \,.\, (0.05 + 1.05) = 0.1 \,.\, 1.1 = 0.11$

$k_3 = 0.1 \,.\, (0.05 + 1.055) = 0.1 \,.\, 1.105 = 0.1105$

$k_4 = 0.1 \,.\, (0.1 + 1.1105) = 0.1 \,.\, 1.2105 = 0.12105$

Step 2: Final Value

$$y(0.1) = 1 + \frac{1}{6}(0.1 + 2\,.0.11 + 2\,.0.1105 + 0.12105) = 1 + \frac{0.66205}{6} \approx 1.11034$$

3. Accuracy

Local Truncation Error: $O(h^5)$

Global Error: $O(h^4)$

RK4 is much more accurate than Euler's or Modified Euler's method for the same step size.

4. Applications

RK4 is used in:

Physics: motion equations, oscillations

Engineering: control systems, circuit analysis

Biology: population modeling

Finance: modeling interest and investments

Simulations: robotics, computer graphics

It's recommend in practical computation for its balance of simplicity and precision.

5. Advantages and Limitations

Advantages:

High accuracy for moderate step sizes,

Straightforward to implement,

Very applicable to many types of problems.

Limitations:

Requires four function evaluations per step

Not suitable for very stiff equations (implicit methods are better)

Hence This Method is one of the most important and powerful used numerical techniques for solving first-order differential equations. It offers a superb compromise between accuracy, stability, and ease of implementation. For most practical problems that require solving an initial value ODE with high accuracy, RK4 is often the first choice. Its dependability makes it a fundamental method in scientific computing and numerical analysis, and it forms the backbone of many modern simulation tools.

(v)MILNE THOMSON METHOD

The Milne-Thomson method is a technique for making an analytic function from its real or imaginary part. This method was constructed by Louis Melville Milne-Thomson (1937). It is a multi-step predictor-corrector method used for numerically solving first-order ODEs of the form:

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0$$

Unlike single-step methods such as Euler’s Method, Runge-Kutta, Milne’s method uses several previous points to estimate the next value, offering higher accuracy and efficiency. However, it requires starting values, usually obtained using Runge-Kutta (RK4).

1. Multi-step Predictor-Corrector Method

Unlike Euler’s Methid, Runge-Kutta, the Milne method uses multiple previously computed values to estimate the next value. It combines a predictor (based on interpolation) and a corrector (based on Simpson’s rule) for increased accuracy.

2. Milne's Predictor Formula

The predictor formula (Milne’s formula) is:

$$y^{(p)}_{n+1} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n)$$

h: step size

$$f_i = f(x_i, y_i)$$

This formula uses three previous derivatives to predict the next y-value.

3. Milne's Corrector Formula

After the predictor step, the result is improved using the corrector:

$$y^{(c)}_{n+1} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f^{(p)}_{n+1})$$

$$f^{(p)}_{n+1} = f(x_{n+1}, f^{(p)}_{n+1})$$

This corrected value may be re-used in an iterative process to further improve accuracy.

4. Steps in the Milne Method

Step 1: Compute Starting Values

Since Milne’s method needs values at y_0, y_1, y_2, y_3, these are computed using the 4th order Runge-Kutta method.

Step 2: Predict the Next Value

Use the predictor formula to calculate $y^{(p)}_4$.

Step 3: Evaluate Function

Compute $f(x_4, y_4^{(p)})$

Step 4: Correct the Value

Use the corrector formula to compute $y_4^{(p)}$ Optionally, repeat this step for further accuracy.

Step 5: Continue

Slide the window forward and repeat the predictor and corrector for the next steps.

5. Example

Solve $\frac{dy}{dx} = x + y$ with y(0)=1, and h=0.1

Assume we've already computed:

$$y_0 = 1$$

$$y_1 = 1.1103$$

$$y_2 = 1.2428$$

$$y_3 = 1.3997$$

Also, compute:

$$f_0 == 0 + 1 = 1$$

$$f_1 = f(x_1, y_1) = 0.1 + 1.110 = 1.2103$$

$$f_2 = 0.2 + 1.2428 = 1.4428$$

$$f_3 = 0.3 + 1.3997 = 1.6997$$

Predictor (Milne):

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$y_4^{(p)} = 1 + \frac{4 \times 0.1}{3}\{(2 \times 1.2103) - 1.4428 + (2 \times 1.6997)\} \approx 1.5820$$

Corrector (Milne-Simpson):

$$f_4^{(p)} = f(0.4, 1.5820) = 0.4 + 1.5820 = 1.9820$$

$$y_{n+1}^{(c)} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1}^{(p)})$$

$$y_4^{(c)} = 1.2428 + \frac{0.1}{3}(1.4428 + 4(1.6997) + 1.9820) \approx 1.5827$$

This value can be used for further steps.

6. Advantages

High accuracy: More accurate than Euler's Method or Modified Euler's methods.

Efficient: Uses fewer function evaluations per step than Runge-Kutta.

Predictor-corrector: Combines speed (predictor) with accuracy (corrector).

7. Limitations

Not self-starting: Requires several initial values computed via other methods.

Needs prior computed values,

Less stable for stiff equation,

Can become unstable if step size h is too large.

Error accumulation: If previous values are inaccurate, error may propagate.

8. Applications

Engineering simulations: heat, fluids, structures

Physics modeling: motion, decay

Economics: growth models

Biology: population dynamics.

Its strength belong in repeated solution of ODEs where past points are reused, saving time while maintaining accuracy.

The Milne-Thomson method is a powerful and important tool in the numerical analyst's toolkit. By combining previously calculated data and applying both prediction and correction, it achieves a balance of computational efficiency and numerical accuracy. It’s best for the problems where a long interval must be traversed with a relatively small step size and high precision.

4.3 REAL LIFE MODELING EXAMPLES

Ordinary Differential Equations (ODEs), especially of the first order, are used to model a variety of dynamic systems in real life. These models gives how a quantity changes with respect to another (often time) and are convenient in fields such as biology, economics, physics, chemistry, and engineering. The following sections highlight real-life modeling examples where first-order ODEs are applied and solved using standard methods.

(i) Population Growth – Exponential Model (Biology)

Modeling Scenario: In an ideal environment with unlimited resources, population increases proportionally to the current population. This is described by the equation:

$$\frac{dy}{dx} = rP$$

P(t): Population at time

r: Constant rate of growth

t: Time

Method Used: Separable Variables

We can separate and integrate:

$$\frac{1}{P}dP = rdt \Rightarrow \int \frac{1}{P}dP = \int rdt \Rightarrow \ln|P| = rt + C \Rightarrow P(t) = P_0 e^{rt}$$

Real-World Use: This model is used in microbiology to predict bacterial growth in culture and in demography to estimate population trends in rapidly growing regions where resources are abundant.

(ii) Newton's Law of Cooling (Physics/Forensics)

Modeling Scenario:

An object cools down in a room. The rate of change of its temperature is proportional to the difference between the object's temperature and the room temperature:

$$\frac{dT}{dt} = -k(T - T_a)$$

T(t): Temperature of the object at time

T_a: Ambient temperature

k: Positive constant (cooling rate)

Method Used: Linear First-Order ODE

This is a linear equation in standard form:

$$\frac{dT}{dt} + kT = kT_a$$

$$\mu(t)=e^{kt} \Rightarrow T(t) = T_a + (T_0 - T_a)e^{-kt}$$

Real-World Use: Used by forensic scientists to estimate the time of death by measuring the body temperature and ambient conditions.

(iii) Radioactive Decay (Chemistry/Environmental Science)

Modeling Scenario:

The amount of a radioactive substance decreases over time at a rate proportional to the amount remaining:

$$\frac{dN}{dt} = -kN$$

N(t): Amount of substance at time t

k: Decay constant

Method Used: Separable Variables

Separate and integrate:

$$\frac{1}{N}dN = -kdt \Rightarrow \ln N = -kt + C \Rightarrow N(t) = N_0e^{-kt}$$

Real-World Use: Used in dating archaeological finds (carbon dating) and understanding decay in nuclear waste management.

(iv) Investment with Continuous Interest (Economics)

Modeling Scenario: An investment grows continuously at a fixed rate:

$$\frac{dM}{dt} = rM$$

M(t): Money at time t

r: Interest rate

Method Used: Separable Variables

$$\frac{1}{M}dM = rdt \Rightarrow M(t)=M_0e^{rt}$$

Real-World Use: Used in financial modeling to forecast investment returns over time.

(v) Velocity of Falling Object with Air Resistance (Mechanics)

Modeling Scenario:

A falling object experiences gravity and air resistance proportional to its velocity:

$$\frac{dV}{dt} = g - kv$$

v(t): Velocity at time t

g: Acceleration due to gravity

k: Air resistance constant

Method Used: Linear First-Order ODE

Rewrite:

$$\frac{dV}{dt} + kv = g$$

$$v(t) = \frac{g}{k} + (v_0 - \frac{g}{k})e^{-kt}$$

Real-World Use: This model is used in designing parachutes, calculating terminal velocity, studying motion in aerospace and sports science.

(vi) Drug Absorption in the Body (Pharmacokinetics)

Modeling Scenario:

The concentration of a drug in the bloodstream decreases exponentially due to metabolism:

$$\frac{dC}{dt} = -kC$$

Method Used: Separable Variables

$$\int \frac{1}{c}\,dC = \int -kdt \Rightarrow C(t) = C_0e^{-kt}$$

Real-World Use: It helps in determining dosage frequency and predicting how long a drug stays effective in the body.

First-order ODEs play as powerful tools in modeling and solving real-life dynamic problems across various disciplines. The different solution methods—analytical and numerical—help us interpret and predict system behaviors. Whether forecasting population growth, understanding thermal changes, managing investments, or analyzing mechanical motion, first-order differential equations remain foundational in science and engineering.

CHAPTER 5

Method of solution of Partial Differential Equations (PDE’s)

5.1 Introduction

Partial Differential Equations (PDEs) are mathematical equations that involve functions of several variables and their partial derivatives. PDE’s arise naturally in many fields like as physics, engineering, biology, economics, and environmental science. PDE’s are used to describe various phenomena like as heat conduction, wave propagation, fluid flow, and quantum mechanics. It also explain a wide range of physical phenomena including several independent variables and their partial derivatives. These equations are foundational in mathematical modeling of systems involving space and time. Here three key areas where PDEs are widely used include:

Heat Transfer (Parabolic PDEs)

Wave Propagation (Hyperbolic PDEs)

Stress Analysis (Elliptic PDEs)

Each category is linked with specific mathematical structures and solution methods, both analytical and numerical.

5.1 HEAT TRANSFER,WAVE PROPAGATION,STRESS ANALYSIS

(i) Heat Transfer (Parabolic PDEs)

The heat equation gives the conduction of heat through materials:

$$\frac{\partial u}{\partial t} = a\nabla^2 u$$

Solution Methods:

Separation of Variables: Assumes u(x,t)=X(x)T(t). Each part satisfies an ODE after separation. This method is ideal for simple geometries and boundary conditions.

Fourier Series Method: Used when boundary conditions are periodic or finite. The temperature distribution is expressed as a sum of sine and cosine functions.

Finite Difference Method (FDM): Approximates derivatives using differences. Time and space are discretized to compute approximate temperature values at grid points. This is widely used in computational simulations.

Example: For a rod with insulated ends and an initial temperature distribution, the solution using separation of variables gives a Fourier series that describes how heat diffuses over time.

(ii) Wave Propagation (Hyperbolic PDEs)

The wave equation models the vibration of strings, sound waves, and electromagnetic waves:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

Solution Methods:

D'Alembert's Solution (for 1D): $u(x,t) = f(x - ct) + g(x + ct)$

Separation of Variables: Used for bounded domains (e.g., vibrating strings fixed at both ends). Converts PDE to two ODEs.

Finite Difference or Finite Element Methods: Used for numerical simulation in 2D/3D and when geometry and materials are complex.

Example: In a vibrating string with fixed ends, separation of variables gives standing wave solutions involving sine functions in space and cosine/sine in time.

(iii) Stress Analysis (Elliptic PDEs)

In solid mechanics, PDEs describe how stress and strain are divided in a material:

$$\nabla \cdot \boldsymbol{\sigma} + F = 0$$

Solution Methods:

Analytical Techniques: Applicable in simple geometries using Airy stress functions and complex variable methods.

Finite Element Method (FEM): It is the most powerful tool in stress analysis. The domain is divided into elements, and PDEs are solved approximately. It controls complex geometries, varying materials, and loading conditions.

Example: In analyzing a loaded beam, FEM divides the beam into elements and computes stress .Also displacement distributions under external force.

Partial Differential Equations are essential in modeling complex physical phenomena across engineering and science. The method of solution depends on the type of PDE and its physical context:

Application	PDE Type	Solution Methods
Heat Transfer	Parabolic	Separation of Variables,Fourier Series,FDM
Wave Propagation	Hyperbolic	D'Alembert's Solution,Separation of Variables,FDM,FEM
Stress Analysis	Elliptic	Airy Function,FEM,BEM

Analytical methods provide insight into idealized problems, while numerical methods like FEM and FDM are necessary for complex, real-world scenarios. Mastery of these methods is captious for engineers and scientists to simulate and solve practical problems effectively.

5.2 FINITE DIFFERENCE AND FINITE ELEMENT APPLICATIONS

Partial Differential Equations (PDEs) are important to modeling various physical systems such as heat conduction, fluid flow, structural deformation, and electromagnetic fields. Analytical solutions to PDEs are often restricted to simple domains and idealized conditions. Therefore, this methods are widely used, especially in engineering and applied sciences. Among them, the Finite Difference Method (FDM) and Finite Element Method (FEM) are two of the most prominent and powerful approaches.

(i) Finite Difference Method (FDM)

Concept

FDM approximates derivatives in the PDE using difference quotients on a grid of discrete points. The solution domain is divided into a regular mesh, and also derivatives like or replaced by algebraic expressions such as:

$$\frac{du}{dx} \approx \frac{u_{i+1} - u_{i-1}}{2\nabla x}, \frac{d^2u}{dx^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

Applications

Heat Transfer: Solves 1D and 2D heat conduction problems. Time-dependent heat equations are hold using explicit or implicit time-stepping schemes.

Wave Propagation: Simulates vibrating strings, sound waves, and spearmint water waves.

Fluid Dynamics: Applied to solve Navier–Stokes equations in basic geometries.

Groundwater Flow: Models flow through porous media in environmental engineering.

Option Pricing in Finance: Used in Black – Scholes model to value derivatives.

Advantages:

Simple to implement.

Efficient for regular, rectangular domains.

Limitations:

Not suitable for complex geometries or irregular boundaries.

Requires fine grids for high accuracy, lead to large systems.

Less accurate near curved boundaries.

(ii) Finite Element Method (FEM)

Concept

FEM divides the domain into small, simple elements (triangles, quadrilaterals, etc.). Within every element, the solution is approximated by simple functions (polynomials). The PDE is transformed into a weak form and solved over the entire domain by assembling local equations into a global system.

Applications:

Structural Analysis: Stress and strain in beams, bridges, aircraft, etc. (used in civil and mechanical engineering).

Heat Transfer: Handles conduction problems with differing material properties and complex geometries.

Electromagnetic Fields: Solves Maxwell's equations in antennas and circuits.

Biomechanics: Models stress in bones and tissues.

Fluid Flow: Simulates flow in irregular and 3D domains (e.g., blood flow in arteries).

Advantages:

Highly adjustable for complex shapes and boundary conditions,

Accurate for problems with material non-uniformity and differing boundary conditions,

Extensible to 2D and 3D domains.

Limitations:

Mathematically more complex to instrument than FDM.

Requires software (e.g., ANSYS, Abaqus, COMSOL) for large-scale problems.

Requires knowledge of linear algebra and numerical integration.

Both FDM and FEM are important numerical methods for solving partial differential equations in science and engineering:

FDM is best for simple problems with regular shapes and straight forward boundary conditions.

FEM is the selected method for complex geometries, varying materials, and multi-physics simulations.

In modern computational analysis, FEM dominates engineering applications, while FDM remains popular in academic and research settings for its simplicity and scholarly value. Together, they enable the simulation, design, and optimization of systems in every technical field.

CHAPTER 6

DISCUSSION AND CONCLUSION

The application of numerical methods plays a important role in solving real-world problems where analytical methods fail or impractical. This dissertation has explored how numerical techniques such as finite difference, Runge-Kutta methods, Newton-Raphson, and finite element methods serve as vital tools across various domains like engineering, physics, economics, biology, and environmental sciences.

One of the key examination is that numerical methods bridge the gap between theory and practical implementation. For example, differential equations arising in fluid dynamics or heat transfer often have no closed-form solutions. Here, methods such as the finite difference or finite element approach give approximate yet accurate solutions.

Additionally, the study revealed how root-finding algorithms like the Newton-Raphson method are fundamental in optimization, control system analysis, and circuit simulations. Likewise, methods for solving ordinary differential equations, such as Euler's method and Runge-Kutta methods, were found extremely convenient in population modeling, drug diffusion models, and mechanical vibration analysis.

Throughout the study and research, it was also evident that numerical errors such as round-off, truncation, and discretization need to be carefully managed. Proper algorithm selection, step-size control, and convergence analysis are necessary to ensure accuracy and stability.

The integration of numerical methods with modern computing tools has significantly increased their approachability and power. With advances in software such as MATLAB, Python, and ANSYS, even complex multi-dimensional problems can now be solved productively.

This dissertation concludes that numerical methods are essential in modern scientific and engineering analysis. As real-world problems become more complex, the need for accurate and efficient numerical techniques continues to grow. The flexibility and adjustability of numerical methods make them easy for a vast range of applications—from simulating weather patterns and designing aircraft to modeling economic growth or solving biomedical challenges.

The continued improvement of numerical algorithms, alongside increasing computational power, means that these methods will remain a cornerstone of research and industry. To harness their full potential, upcoming work should focus on adaptive methods, error estimation, and hybrid techniques combining numerical and machine learning models. In summary, these methods are not just mathematical tools but enablers of innovation across disciplines. Their application transforms theoretical models into practical solutions, making them an essential component of modern problem-solving.

Bibliography

[1] Burden, R. L., & Faires, J. D. *Numerical Analysis*. Cengage Learning.pub. R. Startton (2011)

[2] Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. *Numerical Recipes: The Art of Scientific Computing (3rd ed.). Cambridge University Press.* (2007).

[3] Zienkiewicz, O. C., & Taylor, R. L.*The Finite Element Method.Elsevier*.(2005)

[4] Incropera, F. P., DeWitt, D. P., Bergman, T. L., & Lavine, A.S. *Fundamentals of Heat and Mass Transfer. Wiley*.(2007)

[5] Versteeg, H. K., & Malalasekera, W. *An Introduction to Computational Fluid Dynamics: The Finite Volume Method. Pearson*.(2007).

[6] Taflove, A., & Hagness, S. C. *Computational Electrodynamics: The Finite-Difference Time-Domain Method*. *Artech House*.(2005)

[7] Murray, J. D. *Mathematical Biology I: An Introduction*. Springer.(2002)

[8] Fernández, M. A., Vidrascu, M., & Figueroa, C. A. "*Medical image-based modeling of the human cardiovascular system using FEM*." Computer Methods

in Applied Mechanics and Engineering, 198(45–46), 3474–3486.(2009)

[9] Judd, K. L. *Numerical Methods in Economics*. *MIT Press*.(1998)

[10] Wilmott, P., Howison, S., & Dewynne, J. *The Mathematics of Financial Derivatives: A Student Introduction*. *Cambridge University Press*.(1995)

[11] Anderson, M. P., & Woessner, W. W. *Applied Groundwater Modeling: Simulation of Flow and Advective Transport*. *Academic Press*.(1992)

[12] C. Edward, Jr *The Historical Development of the Calculas, Springer-Verlag.*7417.(1997)

[13] J. Dongarra, I. Duff, D. Sorensen, and H. van der Vorst *Numerical Linear Algebra for High Performance computers, SIAM Pub.* (1998)

[14] D. Bertsimas and R. Freund Data, Models, and Decisions*: The fundamentals of Management Science, Southwestern College Publishing*. (2000)

[15] K. Atkinson and W. Han *Theoretical Numerical Analysis: A Functional Analysi Framework, Springer Verlag*.(2001)

[16] Research paper on Numerical Methods for Solving First Order Ordinary Differential Equations; M.U. UWAEZUOKE, Department of Mathematics , *Faculty of physical sciences Imo State University*. P.m.b 2000, Owerri,

Nigeria.

[17] C. Rathod, Shodhkosh., Application of numerical methods on partial differential equations, *Journal of visual and performing arts* 5(5) ,2024