A COMPARATIVE STUDY OF NUMERICAL METHOD - EULER METHOD & RUNGE – KUTTA METHODS

Dissertation submitted to the Department of Mathematics in fulfillment of the requirement for the award of the degree of Master of Science

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<u>CERTIFICATE</u>

This is to certify that the dissertation entitled “A COMPARATIVE STUDY OF NUMERICAL METHOD - EULER METHOD & RUNGE – KUTTA METHODS” submitted to the Department of Mathematics (MSSV), NAGAON, ASSAM, in partial fulfillment of the requirements for the award of the Degree of M.Sc. in Mathematics is a record of original research work done by AFSANA NARGISH SULTANA bearing Roll No MAT - 26/23 and Reg. No MSSV- 0023-101-001349 in the MAHAPURUSHA SRIMANTA SANKARADEVA VISWAVIDYALA, NAGAON, under my guidance and supervision.

To the best of my knowledge, the matter embodied in this dissertation has not been submitted to any other University / Institute for the award of any degree or diploma.

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DECLERATION

I hereby declare that the dissertation entitled “A COMPARATIVE STUDY OF NUMERICAL METHOD - EULER METHOD & RUNGE – KUTTA METHODS”. Submitted in partial fulfillment of the requirements for the award of the degree of Master of Science in Mathematics, in the discipline of mathematics department, MSSV, NAGAON, is my original work and has not been submitted earlier, either in part or full, to any other University or institution for the award of any degree or diploma.

I further declare that the work presented in this dissertation is entirely my own and has been carried out under the supervision and guidance of DR. MIRA DAS, Assistant Professor, Department of Mathematics, MAHAPURUSHA SRIMANTA SANKARADEVA VISWAVIDYALA NAGAON.

I have duly acknowledged all the sources of information used in the preparation of this dissertation.

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ABSTRACT

The Numerical solution of Ordinary Differential Equations (ODE'S) plays a critical role in various scientific and engineering applications where analytical solutions are either difficult or impossible to obtain. This dissertation presents a comprehensive study of two fundamental numerical techniques for solving initial value problems of ordinary differential equations: the **Euler Method** and the **Runge-Kutta Methods**.

Chapter- 1, in this chapter we discuss about the basic concept of numerical method.

Chapter- 2, in this chapter we discuss about the preliminaries of numerical method.

Chapter- 3, in this chapter we discuss about the **Euler Method,** it's Derivation, Advantages and Disadvantages of **Euler Method**.

Chapter- 4, in this chapter we discuss about the **Runge-Kutta Methods**, it's Derivation, Advantages and Disadvantages of **Runge-Kutta Methods**.

Chapter- 5, in this chapter we solve a same problem by different two methods- **Euler Method** and **Runge-Kutta Methods**, and discuss their results, their applications in real life.

Chapter- 6, provides us the conclusions of this dissertation.

Finally, recommendations for future research and a list of few further works have been mentioned.

LITERATURE REVIEEW

Numerical methods for solving ordinary differential equations (ODEs) play a critical role in science, engineering, and applied mathematics. Among these, the Euler Method and the Runge-Kutta Method are two of the most fundamental and widely used approaches. Over the years, extensive research has been conducted to analyze, compare, and improve these methods, focusing on their accuracy, stability, and computational efficiency.

EULER METHOD-

The Euler Method, named after Leonhard Euler, is one of the simplest numerical techniques for solving initial value problems of ODEs. It is a first order method that approximates the solution by iteratively using the slope of the function at known points.

The Euler Method was introduced in the 18th century by Euler as a means to approximate the solutions of differential equations. Although simple, it laid the groundwork for modern numerical analysis and inspired the development of more sophisticated techniques.

Numerous studies have evaluated the effectiveness and limitations of the Euler method. According to Butcher (2008), while the Euler method is easy to implement, its low accuracy and conditional stability make it less suitable for stiff or complex problems. Researchers such as Iserles (2009) emphasized that its global truncation error is of order O(h), which limits its use for high- precision applications.

Several studies, including those by Hairer et al. (1993), have explored modifications of the Euler method, such as the Improved Euler Method (Heun’s method) and Midpoint Methods, to enhance accuracy without significantly increasing computational cost.

RUNGE-KUTTA METHODS-

The Runge-Kutta (RK) methods, developed by Carl Runge and Wilhelm Kutta in the early 20^{th} century, are a family of iterative techniques designed to overcome the limitations of the Euler Method. The most popular among these is the classical Fourth- order Runge-Kutta method (RK4).

Carl Runge (1895) and Wilhelm Kutta (1901) independently contributed to the development of higher- order methods for solving ODEs. Their work led to methods that offer significantly improved accuracy while maintaining simplicity in implementation.

The RK methods have been extensively studied for their superior performance compared to first- order methods. According to Butcher (2008) and Hairer et al. (1993), the RK4 method, in particular, achieves a global truncation error of O(h^4), making it highly accurate for a wide range of problems.

Research by Dormand and Prince (1980) introduced adaptive step- size control in RK methods, significantly improving their efficiency and applicability to complex real- world problems. Further studies have extended RK methods to systems of equations, stiff problems, and partial differential equations.

Recent research, such as by Verner (2010), has focused on optimizing Rk methods for parallel computing and reducing computational overhead in large- scales simulations.

The literature clearly establishes the Euler method as a foundational but limited technique, primarily useful for educational purposes and simple problems. The Runge-Kutta methods, especially RK4, have become the standard in numerical ODE solving due to their higher accuracy and stability. Ongoing research continues to improve these methods, addressing challenges related to stiffness, computational efficiency, and large- scale problem- solving.

CONTENTS

CHAPTER-1

BASIC CONCEPT OF NUMERICAL METHOD

1.1 INTRODUCTION

Numerical Methods are techniques that are use approximations and algorithm to solve Mathematical problems, especially when exact solutions are difficult or impossible to obtain using traditional algebraic methods. Also it provides approximate solutions rather than exact solutions especially when analytical solutions are not feasible.

- Numerical Methods are widely used in various fields like Engineering, Science and Computer Science, where they help solve problem involving equations, differential equations and other Mathematical procedures.
- Numerical Methods reduce the solution of Mathematical problems to computations that can be performed manually or by means of calculating machines. The development of new Numerical methods are use in Computer have to lead to the rise of Computer Mathematics.
- Numerical Methods designed for the constructive solution of Mathematical problems requiring particular Numerical Results, usually on a computer. A Numerical Method is complete set procedures for the solution of a problem, together with a computable error estimate. The study and implementation of such methods is the province of Numerical Analysis.

Numerical methods naturally find applications in all fields of Engineering and physical sciences, but in the 21st century, the life sciences and even the arts have adopted elements of scientific computations, ordinary differential equations appear in the moment of heavely bodies (plants, stars and galaxies), optimization occurs in portfolio management. Numerical linear algebra is important for data analysis, stochastic differential equations and Markov chains are essential in simulating living cells for medicine and biology.

1.2 CONTRIBUTIONS OF NUMERICAL METHODS

The overall goal of Numerical Methods is the design and analysis of techniques to give approximate but accurate solutions to hard problems, the variety of which are given below-

a) Advanced Numerical Methods are essential in making Numerical weather prediction feasible.
b) Computing the trajectory of a spacecraft requires the accurate numerical solution of a system of ordinary differential equations.
c) Insurance companies use numerical programs for actuarial analysis. All of the above requires a better technique which will minimize the computation error.

There are several methods for solving differential equations having numerical co-efficient with initial or boundary conditions. Some methods will be discussed in the next chapters.

CHAPTER- 2

PRELIMINARIES

EULER METHOD-

2.1 INTRODUCTION

The Euler Method is one of the simplest and most fundamental numerical techniques for solving ordinary differential equations with a given initial value. It provides an approximate solution by using the derivatives value at a known point to estimate the value at the next point.

2.2 BASIC IDEA

If you know the value of a function at a point and the slope at that point, you can estimate the value of the function a short distance away using a straight line approximation.

2.3 INITIAL VALUE PROBLEM

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

Euler's method estimates the next value as,

$$y_{n+1} = y_0 + h \cdot f(x_0, y_n)$$

$h =$ Step size, $(x_n, y_n) =$ current point, $f(x_n, y_n) =$ Slope at the point.

RUNGE-KUTTA METHOD-

2.4 INTRODUCTION

The Runge-Kutta methods are a family of iterative techniques used to obtain numerical solutions of ordinary differential equations. These methods improve upon the Euler method by using intermediate steps to achieve better accuracy significantly reducing the step size.

2.5 BASIC IDEA

Instead of using just the slope at the beginning of an interval, Runge-Kutta methods take a weighted average of stapes at multiple points within the interval. This gives a more accurate estimate of the solution at the next step.

2.6 INITIAL VALUE PROBLEM

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

In second order Runge-Kutta,

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$\Delta y = \frac{1}{2}(k_1 + k_2)$$

Therefore, $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

In fourth order Runge-Kutta method,

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + {}^{h}/_{2}, y_n + {}^{h}/_{2}\, k_1\right)$$

$$k_3 = f\left(x_n + {}^{h}/_{2}, y_n + {}^{h}/_{2}\, k_2\right)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

Therefore,

$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$

CHAPTER- 3

EULER METHOD-

3.1 INTRODUCTION

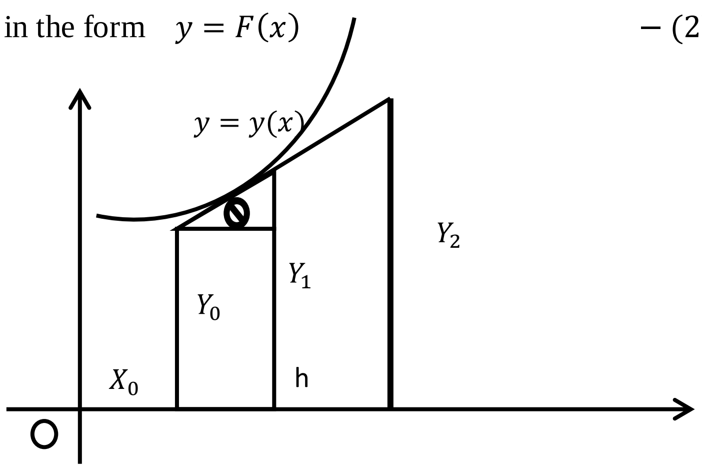
In Mathematics the Euler Method is a first-order numerical procedure for solving ordinary differential equations with a given initial value. It is the most basic explicit method for numerical integration of ordinary differential equations and is the simplest Runge-Kutta method. The Euler method is Name after 'Leonhard Euler', who first proposed it in his book Institutionum calculi integrals (published 1768-1770).

3.2 DERIVATION

Let the differential equation be

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \qquad -(1)$$

Now, we integrating the equation (1) we get a relation between y and x which can be written in the form $y = F(x)$ $\qquad -(2)$



In xy-plan, the equation (2) represents a curve practically a smooth curve is straight for a short distance from any point on it. Hence we have the approximate relation-

$$\Delta y \cong \Delta x tan\theta$$

$$(y_1 - y_0) \cong \Delta x (\frac{dy}{dx})_0$$

$$\cong \Delta x f(x_0, y_0)$$

$$\therefore \quad y_1 \cong y_0 + \Delta x \cdot f(x_0, y_0)$$

$$y_1 \cong \Delta y_0 + hf(x_0, y_0)$$

Then y_1 is the approximate value of y, $x = x_0$, similarly the value of y corresponding to $x_2 = x_1 + h$, $x_{3=}x_2 + h$, … etc are given by

$$y_2 \cong y_1 + hf(x_1, y_1)$$

$$y_3 \cong y_2 + hf(x_2, y_2)$$

In general we obtain,

$$y_{n+1} \cong y_n + hf(x_n, y_n) \quad , n = 0,1,2, \ldots \qquad - (3)$$

Taking h small enough and continuing in their way we could get the integral of (1) as a set of corresponding values of x and y.

This process is very slow. For practical use the method is unsuitable because to get reasonable accuracy with the method we need to give a comparatively smaller value to h. if h is not small then method is too inaccurate.

3.3 ADVANTAGES AND DISADVANTAGES OF EULER METHOD

ADVANTAGES

Since in this method no integration appeared in calculation, this is easier than the Taylor's and Picard's method. Here are its main advantages:

i. Each step requires just one function evaluation and a simple addition, making it computationally cheap for small problem.
ii. It helps to understand the basic idea of numerical integration and how differential equations are solved approximately.
iii. Even if not very accurate, it can provide a quick, rough estimate that can be improved by more advanced methods later.
iv. Can be adapted easily to systems of equations and modified.
v. It's very easy to understand and implement, even by hand or with basic coding.

DISADVANTAGES

i. For stiff equations or sensitive systems, Euler's method can become unstable unless extremely small step sizes are used.
ii. To get reasonable accuracy, very small steps are needed, which increases the number of calculations and slow things down.
iii. The total error grows linearly with time, making it unreliable for long time simulations.

CHAPTER- 4

RUNGE-KUTTA METHODS

4.1 INTRODUCTION

The Runge-Kutta method is family of numerical methods used to approximate solutions of initial value problems for ordinary differential equations. It's particularly popular for its accuracy and stability, especially the fourth-order Runge-Kutta method.

The Runge-Kutta method was developed by two German men Carl Runge (1856-1927) and Martin Kutta (1867-1944) in 1901.

The First (1^{st}) order Runge-Kutta Methods also known as "Euler's Method". This is the simplest Runge-Kutta method. It approximates the solution at the next time step using the slope at the current time step. The error in the 1^{st} order Runge-Kutta formula is h^2.

The Second (2^{nd}) order Runge-Kutta methods use two slope evaluations per step, improving accuracy over Euler's method. Examples include the midpoint method and the improved Euler method. The error in the 2^{nd} order Runge-Kutta formulae is h^3.

The Third (3^{rd}) order Runge-Kutta methods use three slope evaluations per step, offering higher accuracy. The error in the 3^{rd} order Runge-Kutta formula is h^4.

The Fourth (4^{th}) order Runge-Kutta Methods used to offering a good balance between accuracy and computational cost. It uses four slope evaluations per step. The error in the 4^{th} order Runge-Kutta formulae is h^5.

4.2 DERIVATION OF SECOND ORDER RUNGE-KUTTA METHOD

Consider the differential equation, $\frac{dy}{dx} = f(x, y)$ or $y' = f(x, y)$, with the initial condition, $y(x_0) = y_0$. Let h be the interval between equidistance values of x. then in the Second order Runge-Kutta formulae the 1^{st} increment in y is computed from the formulae –

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$\Delta y = \frac{1}{2}(k_1 + k_2)$$

Taken in the given order,

Then, $x_1 = x_0 + h$ and

$$y_1 = y_0 + \Delta y$$

$$= y_0 + \frac{1}{2}(k_1 + k_2)$$

In similar manner the increment in y for the 2^{nd} interval is computed by means of the formulae,

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf(x_1 + h, y_1 + k_1)$$

$$\Delta y = \frac{1}{2}(k_1 + k_2)$$

Therefore, $x_2 = x_1 + h$

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$

And similar method for $x_3$and y_3. The error in the Second order Runge-Kutta is of order h^3.

4.3 DERIVATION OF FOURTH ORDER RUNGE-KUTTA METHOD

Consider the differential equation $\frac{dy}{dx} = y' = f(x, y)$ with initial condition $y(x_0) = y_0$.

Let h be the interval of equidistance values of x. then the 1st increment of y is computed by the formulae –

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Then $x_1 = x_0 + h,$

$$y_1 = y_0 + \Delta y$$

In similar, for next increment in *y* for next interval is computed by the formula –

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$$

$$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$$

$$k_4 = hf(k_1 + h, y_1 + k_3)$$

And$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ taken in given order.

Then$x_2 = x_1 + h,$

$$y_2 = y_1 + \Delta y$$

Similarly, for next interval, the error in the Fourth order Runge-Kutta formulae is h^5.

4.4 ADVANTAGES AND DISADVANTAGES OF RUNGE-KUTTA METHODS

ADVANTAGES - The Runge-Kutta methods offer several advantages when solving ordinary differential equations, particularly initial value problem. Here we discuss various advantages of Runge-Kutta methods –

i. They can be easily applied to a wide range of problems, including stiff and non-linear equations.
ii. Higher-order Runge-Kutta methods offer better numerical stability compared to low-order methods.
iii. The 4^{th} order Runge-Kutta method is a scientific computing due to its balance between efficiency and accuracy.
iv. Unlike Taylor series methods, they do not require computation of higher-order derivatives, only evaluations of the function.
v. Compared to similar methods like Euler's method, Runge-Kutta methods especially the 4th order Runge-Kutta method provide significantly better accuracy with fewer steps

DISADVANTAGES

Here are some disadvantages of Runge-Kutta methods discuss below-

i. Higher order Runge-Kutta Methods requires multiple function evaluations per step, making them more computationally expensive than simpler methods like Euler's.
ii. Traditional Runge-Kutta methods use a fixed step size, which can lead to inefficiency small steps are needed for accuracy, but large steps are faster.
iii. They require storing intermediate function evaluations, which can increase memory requirements, especially in large systems.

CHAPTER- 5

SOLUTION OF SAME PROBLEM BY EULER METHOD AND RUNGE-KUTTA METHODS

5.1 PROBLEM:

Solve, $\frac{dy}{dx} = x + y^2, y = 1$ when $x = 0$ for values at $x = 0$ to $x = 0.4$ taking $h = 0.1$

SOLUTION:

Given, $\frac{dy}{dx} = x + y^2$ and $h = 0.1, x_0 = 0, y_0 = 1$

We have, $x_0 = 0$

$$x_1 = x_0 + h$$

$$x_1 = 0 + 0.1$$

$$= 0$$

$$x_2 = x_1 + h$$

$$x_2 = 0.1 + 0.1$$

$$= 0.2$$

$$x_3 = x_2 + h$$

$$x_3 = 0.2 + 0.1$$

$$= 0.3$$

$$x_4 = x_3 + h$$

$$= 0.3 + 0.1$$

$$= 0.4$$

Now by Euler's numerical method-

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$\therefore y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = 1 + 0.1 \times (0 + 1)$$

$$= 1.1$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.1 + 0.1 \times [0.1 + (1.1)^2]$$

$$= 1.231$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= 1.231 + 0.1 \times [0.2 + (1.231)^2]$$

$$= 1.4025361$$

$$y_4 = y_3 + hf(x_3, y_3)$$

$$= 1.4025361 + 0.1 \times [0.3 + (1.4025361)^2]$$

$$= 1.629246$$

Therefore,

For $x_0 = 0, \quad y_0 = 1$

$$x_1 = 0.1, \quad y_1 = 1.1$$

$$x_2 = 0.2, \quad y_2 = 1.231$$

$$x_3 = 0.3, \quad y_3 = 1.4025361$$

$$x_4 = 0.4, \quad y_4 = 1.629246$$

By 2nd order Runge-Kutta method, we have

Given, $\frac{dy}{dx} = x + y^2$, and $\quad h = 0.1, x_0 = 0, y_0 = 1$

Here, $k_1 = hf(x_0, y_0)$

$$= 0.1 \times [0 + 1^2]$$

$$= 0.1 \times 1$$

$$= 0.1$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$= 0.1 \times [0.1 + (1.1)^2]$$

$$= 0.1 \times 1.31$$

$$= 0.131$$

$$\therefore \quad \Delta y = \frac{1}{2} \times (k_1 + k_2)$$

$$= \frac{1}{2} \times (0.1 + 0.131)$$

$$= \frac{1}{2} \times 0.231$$

$$= 0.1155$$

$$\therefore \quad x_1 = x_0 + h$$

$$= 0 + 0.1$$

$$= 0.1$$

$$y_1 = y_0 + \Delta y$$

$$= 1 + 0.1155$$

$$= 1.1155$$

Again, $k_1 = hf(x_1, y_1)$

$$= 0.1 \times [0.1 + (1.1155)^2]$$

$$= 0.1 \times 1.34434025$$

$$= 0.134434$$

$$k_2 = hf(x_1 + h, y_1 + k_1)$$

$$= 0.1 \times [0.2 + (1.249934)^2]$$

$$= 0.176233$$

$$\therefore \qquad \Delta y = \frac{1}{2}(k_1 + k_2)$$

$$= \frac{1}{2}[0.134434 + 0.176233]$$

$$= \frac{1}{2} \times 0.310667$$

$$= 0.1553335$$

$$\therefore \qquad x_2 = x_1 + h$$

$$= 0.1 + 0.1$$

$$= 0.2$$

$$y_2 = y_1 + \Delta y$$

$$= 1.1155 + 0.1553335$$

$$= 1.2708335$$

Again, $\qquad k_1 = hf(x_2, y_2)$

$$= 0.1 \times [0.2 + (1.2708335)^2]$$

$$= 0.1 \times 1.8150177$$

$$= 0.18150177$$

$$k_2 = hf(x_2 + h, y_2 + k_1)$$

$$= 0.1 \times [0.3 + 1.9747]$$

$$= 0.1 \times 2.27476$$

$$= 0.2274766$$

$$\therefore \quad \Delta y = \frac{1}{2}(k_1 + k_2)$$

$$= \frac{1}{2} \times [0.18150177 + 0.2274766]$$

$$= 0.204489$$

$$x_3 = x_2 + h$$

$$= 0.2 + 0.1$$

$$= 0.3$$

$$\therefore \quad y_3 = y_2 + \Delta y$$

$$= 1.2708335 + 0.204489$$

$$= 1.475322$$

Again, $k_1 = hf(x_3, y_3)$

$$= 0.1 \times [0.3 + (1.475322)^2]$$

$$= 0.1 \times 2.476575$$

$$= 0.24765$$

$$k_2 = hf(x_3 + h, y_3 + k_1)$$

$$= 0.1 \times [0.4 + (1.72297)^2]$$

$$= 0.1 \times 3.368625$$

$$= 0.336862$$

$$\therefore \qquad \Delta y = \frac{1}{2}(k_1 + k_2)$$

$$= \frac{1}{2} \times [0.24765 + 0.336862]$$

$$= 0.292256$$

$$\therefore \qquad y_4 = y_3 + \Delta y$$

$$= 1.475322 + 0.292256$$

$$= 1.767578$$

Therefore, $x_0 = 0, \quad y_0 = 1$

$$x_1 = 0.1, \quad y_1 = 1.1155$$

$$x_2 = 0.2, \quad y_2 = 1.2708335$$

$$x_3 = 0.3, \quad y_3 = 1.475322$$

$$x_4 = 0.4, \quad y_4 = 1.767578$$

Again solve by RK4 order method, we have

Given, $\frac{dy}{dx} = f(x, y) = x + y^2$ and $h = 0.1,\ x_0 = 0,\ y_0 = 1$

Then by RK4 order formulae –

$$k_1 = hf(x_0, y_0) = h(x_0 + y^2{}_0)$$

$$= 0.1 \times [0 + 1^2]$$

$$= 0.1$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$= 0.1 \times [0.05 + (1.05)^2]$$

$$= 0.11525$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$$

$$= 0.1 \times [0.05 + (1.057625)^2]$$

$$= 0.116857$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1 \times [0.1 + (1.116857)^2$$

$$= 0.134736$$

$$\therefore\ \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$=\frac{1}{6}[0.1+2\times0.11525+2\times0.116857+0.134736]$$

$$=\frac{1}{6}\times0.698950$$

$$=0.116491 \therefore \quad x_1=x_0+h$$

$$=0+0.1$$

$$=0.1$$

$$y_1=y_0+\Delta y$$

$$=1+0.116491$$

$$=1.116491$$

Again, $k_1=hf(x_1,y_1)$

$$=0.1\times[0.1+(1.116491)^2]$$

$$=0.13465521$$

$$k_2=hf(x_1+\frac{h}{2},y_1+\frac{k_1}{2})$$

$$=0.1\times[0.15+(1.18381)^2]$$

$$=0.15514$$

$$k_3=hf(x_1+\frac{h}{2},y_1+\frac{k_2}{2})$$

$$=0.1\times[0.15+(1.194061)^2]$$

$$= 0.1575781$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1 \times [0.2 + (1.274069)^2]$$

$$= 0.182325$$

$$\therefore \quad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.15706943$$

$$\therefore \quad x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

And $\quad y_2 = y_1 + \Delta y = 1.27356$

Again $\quad k_1 = hf(x_2, y_2)$

$$= 0.1 \times [0.2 + (1.27365)^2]$$

$$= 0.1821955$$

$$k_2 = hf(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2})$$

$$= 0.1 \times [0.25 + (1.3646577)^2]$$

$$= 0.221229$$

$$k_3 = hf(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2})$$

$$= 0.1 \times [0.25 + (1.384174)^2]$$

$$= 0.21659$$

$$k_4 = hf(x_2 + h, y_2 + k_3)$$

$$= 0.1 \times [0.3 + (1.4901539)^2]$$

$$= 0.2520558$$

$$\therefore \quad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.2183$$

$$\therefore \quad x_3 = x_2 + h = 0.3$$

$$y_3 = y_2 + \Delta y$$

$$= 1.491876$$

Again, $k_1 = hf(x_3, y_3)$

$$= 0.1 \times [0.3 + (1.491876)^2]$$

$$= 0.2525694$$

$$k_2 = hf(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2})$$

$$= 0.1 \times [0.35 + (1.6181607)^2]$$

$$= 0.296844$$

$$k_3 = hf(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2})$$

$$= 0.1 \times [0.35 + (1.64029)^2]$$

$$= 0.304057$$

$$k_4 = hf(x_3 + h, y_3 + k_3)$$

$$= 0.1 \times [0.4 + (1.795933)^2$$

$$= 0.362537$$

$$\therefore \qquad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.302818$$

$$\therefore \qquad x_4 = x_3 + h = 0.4$$

$$y_4 = y_3 + \Delta y$$

$$= 1.491876 + 0.302818$$

$$= 1.794694$$

Therefore, $x_0 = 0, \quad y_0 = 1$

$$x_1 = 0.1, \quad y_1 = 1.116491$$

$$x_2 = 0.2, \quad y_2 = 1.273565$$

$$x_3 = 0.3, \quad y_3 = 1.491876$$

$$x_4 = 0.4, \quad y_4 = 1.794694$$

5.2 COMPARATIVE ANALYSIS

Criteria	Euler method	RK methods
Order of Accuracy	First order	Fourth order
Stability	Low	High
Computation time	Low	Moderate
Complexity	Simple	Complex
Error Accumulation	High	Low

5.3 RESULTS AND DISCUSSION

Graphical comparison of Numerical VS exact solutions for chosen test cases. Error tables and convergence analysis show that Runge-Kutta methods significantly outperform Euler for smaller step sizes and long term integration.

5.4 APPLICATIONS OF EULER METHODS AND RUNGE-KUTTA METHODS

Euler method and Runge-Kutta methods are widely used numerical techniques to solve ordinary differential equations (ODEs) where analytical solutions are difficult or impossible to obtain. Below are the key applications of both methods:

APPLICATIONS OF EULER METHOD

A. Engineering Simulations

- Modeling electrical circuits (e.g., RC, RL circuits)
- Structural analysis problems with dynamic responses
- Fluid flow and heat transfer simulations

B. Physics

- Motion of objects under the influence of forces (kinematics and dynamics)
- Simple harmonic motion problems
- Planetary motion approximations

C. Biological and Ecological Modeling

- Population growth models (e.g., logistic growth)
- Spread of infectious diseases (basic epidemic models)

D. Economics

- Modeling investment growth
- Basic economic dynamic models (e.g., supply-demand variations over time)

E. Control Systems

- Simulating system responses to control inputs
- Stability analysis in control Engineering

F. Education and Learning

- Useful for understanding the basics of numerical approximation
- Often used as the first numerical method taught to students.

APPLICATIONS OF RUNGE-KUTTA METHODS

The RK methods, especially the 4^{th} order Runge-Kutta method are more accurate and stable compared to Euler's method, making them suitable for:

I. Engineering
   - Complex dynamic simulations in mechanical and aerospace engineering
   - Vibration analysis and structural dynamics
   - Electrical circuit simulations with higher accuracy.

II. Physics
   - Accurate simulation of orbital mechanics
   - Quantum mechanics problems involving time-dependent Schrodinger equation
   - Non-linear dynamical systems

III. Computational Biology
   - Modeling the spread of diseases using SIR, SEIR models
   - Enzyme kinetics and biochemical reaction simulations

IV. Climate and Environmental Modeling
   - Weather prediction models
   - Climate change simulation models

V. Economics and Finance
   - Solving dynamic economic models with non-linear differential equations

- Financial forecasting models

VI. Robotics and control Systems

- Path planning and trajectory prediction
- Simulation of autonomous system dynamics

VII. Computer Graphics and Animation

- Simulating realistic object motion
- Physics-based animation of practices and fluids.

CHAPTER- 6

CONCLUSION

Both the Euler method and Runge-Kutta methods are fundamental tools for numerically solving ordinary differential equations, but they differ significantly in terms of accuracy and efficiency.

EULER METHOD, is simple and easy to implement, making it suitable for problems where high accuracy is not essential or for educational purposes. However, it is less accurate and can accumulate significant error, especially for larger step sizes.

RUNGE-KUTTA METHODS, particularly the fourth order Runge-Kutta, offer much higher accuracy and stability, even with moderately large step sizes. They are widely used in practical application because they balance computational cost with precision.

In summary, while Euler's method is best for simple or illustrative problems, Runge-Kutta methods are preferred for most real-world applications due to their superior accuracy and reliability.

REFERENCES

- Dormand, J.R., & Prince, P. J. (1980). “A family of embedded Runge-Kutta formulae.” *Journal of Computational and Applied Mathematics,* 6(1), 19-26.
- Lambert, J. D. (1991). *Numerical Methods for Ordinary Differential Systems: The Initial Value Problem.* John Wiley & Sons.
- Hairer, E., Norsett, S.P., & Wanner, G.(1993). *Solving Ordinary Differential Equations* /: *Nonstiff Problems* (2nd ed.). Springer.
- Smith, H.L. (2005). *An Introduction to Delay Differential Equations with Applications to the Life Sciences.* Springer.
- Butcher, J.C. (2008). *Numerical Methods for Ordinary Differential Equations* (2nd ed.). John Wiley & Sons.
- Iserles, A. (2009). *A First Course in the Numerical Analysis of Differential Equations* (2nd ed.). Cambridge University Press.
- Verner, J.H. (2010). “Efficient Runge-Kutta pairs for solving ODEs.” *SIAM Journal on Numerical Analysis,* 47(1), 236-249.
- Burden, R. L., & Faires, J. D. (2011). *Numerical Analysis* (9th ed.). brooks/ Cole, Cengage Learning.

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