THE APPLICATION OF THE AXIOM SCHEME IN COMPREHENDING DIFFERENT CONCEPTS OF ABSTRACT ALGEBRA

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Certificate

This is to certify that **DYUTTI ANVESHA SARMA** bearing **Roll No MAT 25/23** and **Regd. No. MSSV-0023-101-001345** has prepared her dissertation entitled "**THE APPLICATION OF THE AXIOM SCHEME IN COMPREHENDING DIFFERENT CONCEPTS OF ABSTRACT ALGEBRA**" submitted to the Department of Mathematics, **MAHAPURUSHA SRIMANTA SANKARADEVA VISWAVIDYALAYA,** Nagaon, for fulfillment of MSc. degree, under guidance of me and neither the dissertation nor any part thereof has submitted to this or any other university for a research degree or diploma.

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DECLARATION

I Dyutti Anvesha Sarma bearing the Roll No – MAT- 25/23, hereby declare that this dissertation entitled, "The Application Of The Axiom Scheme In Comprehending Different Concepts of Abstract Algebra" was carried out by me under the supervision of my guide Dr. Maitrayee Chowdhury, assistant professor, Mahapurusha Srimanta Sankaradeva Viswavidyalaya, Nagaon. The study and recommendation drawn are original and correct to the best of my knowledge.

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Table of Contents

ABSTRACT

Mathematical logic, as described by A.G. Hamilton, a renowned mathematician, is the exploration of logical systems and their interpretation, primarily focusing on the formal systems and their rules to express and manipulate relationships within the grasp of logical reasoning. It is a branch of mathematical science that applies the formal methods to investigate the groundwork for reasoning and proof. George Boole, through his work "Mathematical Analysis of Logic", laid the foundation for what is today known as sentential logic. Logical perception entails understanding how to comprehend and reason in a mathematically structured manner. One of most basic topics in logical analysis is to grasp the fundamentals of propositional calculus, which includes the concepts of simple statements, truth values (T or F) interpretation and logical connectives. Propositions cannot be simultaneously true and false. Logical connectives or operators that are commonly employed to construct compound sentences include negation, conjunction, disjunction and so on. The propositional calculus expanded to encompass the broader understanding of tautology and contradiction. While tautology takes the truth value T under each possible assignment of truth values to the statement variables occur in it, the idea of contradiction can be perceived as the other end of the spectrum. In addition to these two fundamental concepts, the notions of contingency, satisfiability, unsatisfiability, valid and invalid propositions are crucial in the analytic branch of mathematics. The theory of argument and validity is another profound concept one needs to get familiar with to be able to move ahead in the field of logic and reasoning. As per A.G. Hamilton, an argument form is a finite sequence of statement forms, the last of which is regarded as conclusion and the remainder as the premises. An argument can be either valid or invalid.
Our study of logic will be limited without knowing the notion of the formal system which

plays the key role in the analysis of deduction. The idea of formal deductive system comprises of the systematic procedure involving the alphabets of symbols, set wff, axioms, rules of deduction and ultimately the proof. In a formal system, symbols have no meaning and in dealing with them, we need to be careful to assume nothing about their properties other than what is specified in the system. The formal system, being comprised of chronological sequence of alphabet of symbols, set wff, axioms and rules of deduction and proof , creates the environment for the further development into the first order language. The first order language or first order logic methodically used to describe algebraic structures such as groups, rings and vector spaces. The first order language thus provides a framework for expressing statements about objects, their properties and their inter relationship in a symbolic approach.

1. Introduction

Mathematical logic, as defined by A.G. Hamilton, a prominent mathematician, is the study of the logical systems and their interpretation, with an emphasis on the formal systems that employ symbols and rules to represent and manipulate the connections between the elements of logical analysis. One of the most popular definitions of logic is that it is the analysis of methods of reasoning. It is a meticulous approach to logic, concentrating on the structures and validity of the arguments rather than their specific content. The conspicuous feature of mathematics, as opposed to other sciences, is the use of proofs instead of observations. A physicist may prove physical laws from other physical laws or simply from observations but a mathematician will only accept or come to agreement with an observation once it has been proved. That's when mathematical logic comes into rescue by constructing rigorous proofs, thus ensuring that mathematical statements follow logically from defined axioms and rules of inference. Mathematical logic is a field of mathematics that employ formal techniques to investigate the principles of logical reasoning and proof.

From having it’s very roots in the ancient civilizations like Greece, India, Egypt and China, where early thinkers explored the logical methods, it has now emerged in modern fields in the late 19^{th} century with the work of the mathematicians like Centor, Frege and others, who aimed to establish a sound mathematical analysis.

The history of mathematical logic is an intricate and fascinating journey that connects mathematics, philosophy and computer science, evolving over centuries to become a cornerstone of modern thought. Aristotle (384-322 BCE), also called the founder of formal logic, identified some simple patterns in human reasoning and Leibniz dreamt of the possibility of transforming the reasoning to calculation. He developed the syllogistic logic, a consequential system of the deductive reasoning. The Stoic philosophers also contributed with the propositional logic, while exploring the statements connected by different logical operators (like “and”, “or” and “not”). Medieval scholars in the Islamic world and Europe preserved and expanded upon the Greek logic. As a mathematical discipline, logic is relatively recent: the 19^{th} century pioneers like Bolzano, Boole, Cantor, Dedekind, Frege, Peano are some of the prominent names. From our viewpoint, their work is seen as contributing to Boolean Algebra, set theory, predicate logic, propositional logic, as clarifying the foundations of the natural and real number systems, and introducing the symbolic notation for the logical operations. This period did not only produce some of the most significant theorems in the domain of mathematical logic, but it did contribute to important conceptual advancements and the understanding that the mathematical logic follows clear and explicit mathematical well-set rules.

In the period from 1900-1950, important new ideas regarding logical approach came from Russell, Zermelo, Lusin, Hausdorff, Hilbert, Ramsey, Godel, Tarski, Church and Kleene. They uncovered the initial theorems in the mathematical logic. The early part of the 20th century was also marked by the so-called foundational crisis in mathematics (2). The desire to establish the mathematical logic arose from the efforts made during these times to provide the solid foundation for mathematics. Mathematical logic now has taken a life of its own, and also thrives on many interactions with the other areas of mathematics and computer science. In the second half of the last century, logic as pursued by mathematicians gradually branched into four main areas namely the

Model theory, Recursion theory (or computability theory), set theory and Proof theory. The urge for logic was raised tenfold when mathematicians witnessed the presence of paradoxes, which are the arguments that lead to the contradictions. The mathematical logic thus was further analysed to make it inclusive of Russel's paradox, Cantor's paradox, the liar paradox, Richard's paradox and so on, which further enriched the foundational logical analysis.

The logical perception entails understanding how to comprehend and reason in a mathematically structured manner. One of the most basic concepts in logical analysis is to grasp the fundamentals of propositional calculus, which includes the concepts of simple statements, truth values (T or F) interpretation and logical connectives. Propositions cannot be simultaneously True or False. Propositional calculus starts with the concept of the statement which has a truth value. These statements are connected by the logical connectives or the operators that are commonly employed to construct compound sentences which include negation($\sim$), conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$) and biconditional ($\leftrightarrow$). Negation is one of the simplest most operations found on sentences. Putting negation on a sentence means going for the exact opposite truth value result of the original sentence. Another common truth-functional operation is the conjunction ‘and’. The conjunction of two sentences namely A and B will be designated by A ⋀ B. If we draw the truth table for conjunction, it is clear that A ⋀ B is true if only both A and B are true. In natural languages, there are two distinct usages of ‘or’; the inclusive and exclusive. According to the inclusive usage ‘A’ or ‘B’ means A or B or both, whereas according to the exclusive usage, the meaning is A or B, but not both. We shall use the sign ‘⋁’, for the inclusive connective. If we construct a truth table, then A ⋁ B is false only when both A and B are false. Here A ⋁ B is called disjunction. Learning about conditional operators is another part of propositional calculus. A significant truth-functional operation is the conditional (→): “if A, then B”. This particular operation is designated by A→B. A→B is false only when the antecedent A is true and the consequent B is false. Another operation which we talked about is biconditional ↔. We get the result TRUE under the application of biconditional if both the statements (A and B) are either true or false. The truth values are assigned to the simple sentences, which are then connected by the logical connectives, resulting in the truth values of the conclusive compound sentences. The field of propositional logic expanded to encompass the broader understanding of tautology and contradiction. A statement form is a tautology if it takes the truth value T under each possible assignment of truth values to the statement variables which occurs in it, meanwhile the idea of contradiction can be perceived as the other end of the spectrum (6). A statement form is said to be the contradiction if it takes truth value F under each possible assignment of truth values to the statement variables occurs in it. When we talk about statement form in the domain of logical analysis, the statement form is symbolic representation of a compound statement. It consists of statement variables along with the logical connectives joining them. If A and B are the statement variables, then (∼A), (A⋀B), (A⋁B), (A→B), (A↔B) are the statement forms. If A and B are the two statement forms, A logically implies B if and only if A→B is a tautology and A is logically equivalent to B if and only if A↔B is proven to be a tautology. A compound proposition which is sometimes TRUE and sometimes FALSE, is considered to be contingency. The conditions for satisfiability and unsatisfiability also play key role in the propositional logic. A compound proposition is said to be satisfiable if there is at least one TRUE result in its truth table, otherwise it is considered to be unsatisfiable. A compound statement is also considered valid once it becomes tautology. Tautology is always satisfiable and valid but the satisfiable is not always tautology. On the other hand, contingency is satisfiable despite not being valid (invalid). Contradictions are those statement forms which are neither satisfiable nor valid. If a statement form A and (A→B) are tautologies then B automatically becomes a tautology, which is another significant property of tautology. Some of the most useful tautological implications include: Law of detachment, Modus tollendo ponens, Modus tollendo tollens, Law of simplifications, law of adjustment, law of addition and so on. Apart from understanding tautology, the analytical branch of logic also deals with a concept namely Sentential interpretations and inferences. The sentential interpretation gives meaning (truth value) to the abstract logical formulas. It provides a clear way to inspect and evaluate the truth or falsity of a logical expression based on the assigned values, thereby providing us with the necessary help to understand whether the formula is valid, satisfiable, contingent or a contradiction. Two major criteria which we should adhere to in order to construct the sentential interpretations are:

Criteria 1: Given a set of premises, the rules of logical derivations must permit us to infer only those conclusions which logically follow from the premises.

Criteria 2: Given a set of premises, the rules of logical derivations must allow us to infer all those conclusions which logically follow from the premises.

In case of two statements namely P and Q, the idea is that Q logically follows from P when Q is true in every interpretation for which P is considered to be true. A sentence P is taken as a sentential interpretation of a sentence Q if and only if P can be obtained from Q by replacing the component atomic sentences of the statement Q by other (not necessarily distinct) sentences. In order to be a sentential interpretation of the particular sentence Q, the statement P needs to follow a certain characteristic: Its major sentential connective must be an implication. In other words, a sentential interpretation of a sentence must preserve its sentential form. If a component atomic sentence occurs more than once in a sentence, any sentential interpretation of that particular sentence must replace that component atomic sentence by the same thing in both of its occurrences. Moreover, a statement is said to be valid if under every interpretation where all premises are true, as a result of which conclusion is also true. While writing so many sentential expressions can be tiresome and time consuming, so mathematicians came up with the application of adequate set of connectives. They ensure that the logical expressions and their interpretations can be written with the help of the application of a minimal set of tools. When constructing formal systems (like Hilbert or Natural Deduction systems), using a minimal set makes the system easier to comprehend and analyse. These are essential for evaluating any logical formula or truth function using only a limited, well-chosen set of operations. They are related to the restricted statement forms. A restricted form often simplifies logical expression without the loss of expressiveness. A restricted statement form often depends on an adequate set of connectives for its proper application. To ensure that restricting the statement form does not reduce the expressive capability, we choose to implement the adequate sets of connectives. An adequate (or functionally complete) set of connectives is a minimal group of logical connectives that can be applied to express all the potential truth-functional statement forms. Every possible propositional logic formula can be rewritten by simply using the connectives used in the adequate set. The sets such as $\{\sim, \wedge\}$, $\{\sim, \vee\}$, $\{\sim, \rightarrow\}$, $\{\sim, \wedge, \vee\}$ are some examples of the adequate set of connectives. NOR (indicated by ↓) and NAND (indicated by **l**) are also parts of adequate set of connectives. The singleton sets {↓} and {**l**} are adequate sets of connectives, that means every truth function can be expressed by a statement form in which only ↓ (respectively **l**) appears. By constructing the required truth tables, we can prove our point. That’s how we can come to the conclusion that these two connectives NOR and NAND are functionally complete- meaning they can each independently be used to construct all other propositional logic operators. After discussing the application of the adequate set of connectives, we tend to move ahead to grasp the concept of another pivotal part of mathematical logic, which is the concept of argument and validity (4). An argument form is considered to be a finite sequence of statement forms, that last part of which is regarded as the conclusion and the remaining part as the premises. In mathematical logic, the concepts of argument and validity are foundational to understanding how logical reasoning works. They are central concepts used to evaluate reasoning and determine whether conclusions follow logically from the given set of the premises. An argument is valid if and only if under the condition that the premises are true, the conclusion must also be true. This definition focuses on logical form, not the actual truth of the premises or conclusion. On the other hand, an argument form is considered to be invalid under the condition that if we can assign the truth values to the statement variables occurring in such a way so as to make each of the premises take the truth value T but the conclusion assumes the truth value F. An argument is called sound under the condition that it is valid and all of its premises are true. Propositional calculus along with the adequate connectives and argument forms comprises the informal logic. At this point, we informally observed if the reasoning is correct.

To analyse these argument forms precisely, now we go forward to comprehend a structured concept of the formal system or formal logic in the logical analysis. In the previous logical concepts, we saw how to abstract the forms of statements and arguments in order to see more clearly. Certain questions remained, however for example, can we find a simple procedure which actually enables us to construct an argument and to check its validity. To investigate such questions, we introduced the concept of formal deductive system.

In a formal system, the symbols have no meaning and in dealing with them, we must be careful to assume nothing about their properties other than what is actually specified in the formal system. We initiate by providing a rather general definition of what constitutes a formal system in mathematical logic. For the construction of a formal system, we require the following:

1. An alphabet of symbols.
2. A set of finite strings of those symbols, called the well-formed formulas These are to be thought of as the words and sentences in our formal language system.
3. A set of the well-formed formulas which are also called the axioms. Axioms are statements or the formulas which are assumed to be true without proof. They truly serve as the foundation for all derivations in the system. There are infinitely many axioms in the formal systems.
4. A finite set of 'rules of deduction' i.e. rules which enable us to deduce the well-formed formulas.

One of the most significant systems in mathematical logic is the formal system L of the statement calculus is defined as the following:

- The alphabet of symbols: The set of basic symbols including $\sim, \rightarrow, (\ ,\ ), P_1\ , P_2, P_3,$...
- The well-formed formulas (WFFs)
- The axioms: These are predefined rules or inferences. There are infinite number of axioms so we cannot list them all.

However, we can specify all of them by means of three axiom schemes.

For any wfs, A, B, C the following wffs are the axioms of L,

$(L1)$ $\left(A \rightarrow (B \rightarrow A)\right)$

$(L2)$ $((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

$(L3)$ $(((\sim A) \rightarrow (\sim B)) \rightarrow (B \rightarrow A))$

- Rules of deduction: In the formal system L, there is only one rule of deduction, namely Modus Ponens (MP). It says that from A and $(A \rightarrow B)$, B is the direct consequence, where A and B are any wfs of L.

A proof in a formal system L is a finite sequence of formulas constructed according to the strict syntactic rules with the aim to derive the theorems. Predicate calculus is another key component of logical analysis. It can be perceived as the detailed and structured extension of the propositional calculus and statement logic. While the propositional logic only has to deal with the statement forms, the predicate logic allows us to talk about the internal structuring related to the statement inferences- specifically, objects, properties and the relations in between them. The predicate calculus also introduces us to the existence of quantifiers. Two of the most commonly used quantifiers are the universal quantifier denoted by ∀ (for all) and the existential quantifier denoted by ∃ (there exists). Apart from the concepts of formal systems and axiom schemes, another concept that helped us to dive deeper into formal logical analysis is the concept of First Order Language (FOL). The key ingredients of the first order language are:

1. The variables x_1, x_2, x_3 etc.
2. Individual constants such as $a_1, a_2, a_3, ...$
3. Some of the predicate letters A_i^n
4. Some of the function letters f_i^n
5. The punctuation symbols (,) and the connectives $\sim$ and $\rightarrow$
6. The quantifiers.

The first order language enables the precise formulation of mathematical theorems and proofs, removing the ambiguity found in the natural language. It can describe a wide range of mathematical structures, such as groups, rings, vector space and so on. The first order logic appears to be a natural subject of study, and its discovery is almost certain. It is sufficient to express all the concepts and the principles of ordinary abstract algebra. For a long time, the first order logic has been considered the appropriate logic for the studying of the basis of mathematics. Its emergence is closely tied to the technological advancements, with various perspectives on what defines logic, diverse mathematical research programs and the ongoing philosophical and conceptual discussions. The initial language in mathematical logic signifies a breakthrough in our capacity to formalize and manipulate the abstract reasoning. By enabling the quantification of individual objects and offering a comprehensive framework for expressing properties and relationships, it bridges the gap between the basic propositional logic and the expressive requirements of mathematics. Despite all the limitations and restrictions, its equilibrium between the power and tractability makes it one of the fundamental and extensively used systems in logic, mathematics and computer sciences.

2. Literature Review

The area of Mathematical logic is a fertile one for research work. Many interesting and sophisticated research works in this area have been a source of motivation for many upcoming new researchers. We try to discuss here a few of these articles that have encouraged us a great deal to pursue or dabble with Mathematical logic. L. S. Hay in his work “Axiomatization of the Infinite-valued Predicate Calculus” that is based on the infinite-valued statement calculus tries to give an endeavor and axiomatize the full predicate calculus. For this axiomatization, a property similar to but weaker than completeness is proved (5).

Ruitenberg in his work “Basic Predicate Calculus” establishes a completeness theorem for first order Basic Predicate Logic that is a proper subsystem of intuitionistic predicate logic. Further the author tries to develop the notion of functional well-formed theory as the correct notion over Basic Predicate Logic. For this notion it is seen that strong completeness theorems are possible. Also, the article has the derivation of the undecidability of basic Arithmetic, the basic logic equivalent of intuitionistic Heying Arithmetic and classical Peano Arithmetic (7).

Mathematical Modality: An investigation in higher order logic by Andrew Bacon, Journal of philosophical logic, vol. 53, 131-179, 2024. Andrew Bacon in his work “Mathematical modality: An investigation in Higher -Order Logic” wants to pursue modality in higher order. Through his work, he wants to put emphasis on the contemporary philosophy of mathematics posits and theorizes in terms of the special kinds of mathematical modality. The aim of the article as expressed by the author is to bring into focus some new sophisticated work on modal reality (1).

3. Preliminaries

Definition 3.1. (Statement): A statement is a declarative sentence form which either assumes the value True or False (but not both). It is the building block for the formation of the logical arguments and reasoning procedures.

Definition 3.2. (Truth function): A truth function is a fundamental concept in propositional logic. It refers to the function that actually determines the truth value of a compound statement based solely on the truth values of its component sentential forms.

Definition 3.3. (Connective): In mathematical logic, a connective (also known as logical connective or logical operators) is a symbol or word used to connect more than one statements (propositions) to form a compound sentence whose truth value depends on the truth value function assumed by the individual statement.

Definition 3.4. (Conjunction): A conjunction is a sentential operation. The conjunction of the two statements P and Q (written as $P \wedge Q$) is true only when both the statements P and Q are true.

Definition 3.5. (Disjunction): Another sentential operation frequently used in mathematical logic is disjunction. The disjunction of the two statements P and Q (written as $P \lor Q$) is true if at least one of the statements P or Q is true.

Definition 3.6. (Conditional): In order to form the compound sentences, one of the operators used is the conditional. The conditional denoted by $P \rightarrow Q$, the statement is false if the statement Q is false.

Definition 3.7. (Biconditional): Another operator used in the construction of the compound sentences is biconditional. The biconditional denoted by $P \longleftrightarrow Q$, the statement is false only when P is true and Q is false or vice versa.

Definition 3.8. (Tautology): A statement form is considered as a tautology if it takes the truth value T under each possible assignment of truth values to the statement variables which occur in it.

Definition 3.9. (Contradiction): A statement form is considered as a contradiction if it takes the truth value F under each possible assignment of truth values to the statement variables which occur in it.

Definition 3.10. (Statement form): A statement form is symbolic representation of a compound statement. It consists of statement variables along with logical connectives joining them. If A and B are the statement form, then $(\sim A)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$ are the statement forms.

Definition 3.11. (Logically imply): If A and B are the two statement forms, then A logically implies B if and only if A→B is a tautology.

Definition 3.12. (Logically equivalent): If A and B are the two statement forms, then A is logically equivalent to B if and only if A↔B is a tautology.

Definition 3.13. (Contingency): A compound proposition is said to be contingency if it is sometimes True and sometimes False.

Definition 3.14. (Satisfiable): A compound proposition is satisfiable if there is at- least one True result in its truth value. Hence, we can say that tautology is always satisfiable but the satisfiable is not always tautology.

Definition 3.15. (Unsatisfiable): A compound proposition is unsatisfiable if there is not even a single True result in its truth value.

Definition 3.16. (Valid): A compound proposition is valid when it is a tautology. The valid statements are the key elements in constructing logical interpretations.

Definition 3.17. (Invalid): A compound proposition is invalid when it is either a contradiction or a contingency. It means that if a compound proposition is sometimes True and sometimes False then also it is invalid (4).

Definition 3.18. (Atomic sentence): A sentence is called atomic if it contains no sentential connective in it. In other words, a simple sentence free from any sentential connective is the definition of an atomic sentence. i.e. it is devoid of any presence of ‘or’, ‘and’, ‘conditional’(denoted by →) and so on.

Definition 3.19. (Sentential interpretation)**:** A sentence P is a sentential interpretation of a sentence Q if and only if P can be obtained from Q by replacing the component atomic sentences of Q by the other sentences. A sentential interpretation of a particular sentence must preserve its sentential form under any circumstances. The major sentential connective present in the original sentence should be present in the interpreted sentence. If a component atomic sentence occurs more than once in a sentence, any sentential interpretation of that sentence must replace that component atomic sentence by the same thing in both of its occurrences.

Definition 3.20. (Modus Ponens): One of the most frequently used tautological implication in the study of mathematical logic is the Modus Ponens (MP).

The MP implication is: $(\sim P)\wedge(P\vee Q) \to Q$.

Definition 3.21. (Restricted statement forms): A statement form of the type,

$A = A\,(p, q, \ldots, c = \langle \sim, \wedge, \vee \rangle)$ is said to be the restricted statement form.

Definition 3.22. (Normal forms): Normal forms in the logical interpretations define the standardized or rather precise ways of expressing the logical formulas, making them easier to analyse, comprehend and manipulate- especially in the cases of proof systems. There are two types of normal forms in the mathematical logic namely: Conjunctive Normal Form (CNF) and Disjunctive Normal Form (DNF).

Definition 3.23. (Conjunctive Normal Form or CNF): The conjunctive normal form (CNF) is a standardized way of expressing logical formulas in the propositional calculus. A formula is in CNF if it is a conjunction (denoted by $\wedge$) of one or more than one clause, where each clause is a disjunction (denoted by $\vee$) of atomic sentences. CNF is extremely important in understanding automated theorem proving and the logic programming.

Definition 3.24. (Disjunctive Normal Form or DNF): The disjunctive normal form (DNF) is a standardized way of expressing logical formulas in the propositional calculus. A formula is in DNF if it is a disjunction (denoted by $\vee$) of one or more than one clause, where each clause is a conjunction (denoted by $\wedge$) of atomic sentences. DNF breaks complex expressions into simpler, clearly structured components.

Definition 3.25. (Adequate set of connectives): An adequate set of connectives is a minimal group of logical connectives or mathematical operators that can be applied to write all the potential truth-functional statement forms. Every possible propositional logic formula can be deduced by simply using the connectives used in the adequate set of connectives. The sets such as $\{\sim, \wedge\}$, $\{\sim, \vee\}$, $\{\sim, \rightarrow\}$, $\{\sim, \wedge, \vee\}$ are some of the examples of the adequate set of connectives. NOR (indicated by ↓) and NAND (indicated by **l**) are also parts of adequate set of connectives. We can verify our claim by simply constructing the required truth value tables.

Definition 3.26. (Argument): An argument form is a finite sequence of the statement forms, the last of which is regarded as the conclusions and the remainder as the premises. In mathematical logic, the concepts of argument forms and their validity are foundational to understanding how logical reasoning works. They are key concepts which are used to evaluate analytical reasoning and determine whether conclusions will follow logically from the given set of the premises.

Definition 3.27. (Valid argument form): In the first step, we consider an argument form in the following way,

$A_1, A_2, A_3, \dots, A_n \; ; \; \therefore A$

An argument form is considered to be valid if it is possible to assign the truth values to the statement variables occur in it in such a way so as to make each of the premises $(A_1, A_2, A_3, \ldots, A_n)$ take the truth value T and thus to make the conclusion A take the truth value T.

Definition 3.28. (Invalid argument form): In the first step, we consider an argument form in the following way,

$A_1, A_2, A_3, \ldots, A_n\ ;\ \therefore A$

An argument form is considered to be invalid if it is possible to assign the truth values to the statement variables occur in it in such a way so as to make each of the premises $(A_1, A_2, A_3, \ldots, A_n)$ take the truth value T and to make the conclusion A takes the truth value F.

Definition 3.29. (The formal system): A formal system is the main building block of the deductive logic. It provides us with a precise, rule-dependent environment where the logical reasoning can occur which at the end paves the way for the formulation of the proofs and the theorems. In a formal system, the symbols have no meaning and in dealing with them we need to assume nothing about their properties other than what is specified in the given system. A formal system comprises of the components namely: An alphabet of symbols, a set of finite strings of the symbols which are known as the well-formed formulas (wff), a set of axioms and a finite set of “rules of deduction”.

Definition 3.30. (Well-formed formulas): The well-formed formulas are constructed from the symbols, using the formation rules of the formal system. They do not necessarily need the interpretation to be the well-formed, they are purely syntactic in nature.

Definition 3.31. (Axiom): An axiom is perceived as the fundamental assumption that serves as the building block for the formal system, which are accepted as true to enable the derivation of the additional truths through the precise application of the logical inferences. To establish the rules and framework of a logical or mathematical system, axioms play a crucial role in the further derivation of the proofs and theorems.

Definition 3.32. (Axiom scheme): An axiom scheme is a prototype that exhibits an infinite set of axioms used within a formal system. An axiom scheme signifies a collection of axioms, each obtained by substituting specific formulas into the variables of schemes.

Definition 3.33. (Theorem): In the realm of mathematical logic, a theorem is a statement that has been logically proven by the accurate application of the axioms and the rules of inference within a formal system.

The primary distinction between an axiom and a theorem lies in the fact that a theorem is not primarily assumed to be true which is different for an axiom. For a theorem, its truth is proven through the application of the proof.

Definition 3.34. (Proof): In the field of mathematical logic, a proof is a finite series of logical deductions that ultimately leads to the desired conclusion. A proof can be perceived as a finite sequence of well-formed formulas (wff), which is either :

a. An axiom, or

b. Derived from earlier formulas in the sequence by the application of the rules of inference.

Therefore, we can conclude from above discussion that the proof must logically follow from the axioms or previously proven theorems.

Definition 3.35. (Inference rules): The inference rules are the set of formal logical rules that determine how to derive new formulas (conclusions) from the existing ones within the specified formal system. Modus Ponens (MP) is one of the most important inference rules.

Definition 3.36. (First order language): The first order language is a widely acceptable concept in logical interpretation. A first order language is a formal language utilized with the aim to express the statements about the object and their inter-relationships. A first order language consists of the following:

1. Variables $x_1, x_2, \ldots$

2. Individual constants $a_1, a_2, \ldots$

3. Predicate letters ${A_i}^n$

4. Function letters ${f_i}^n$

5. Punctuation symbols (,) and the connectives like $\sim$ and $\rightarrow$

6. The quantifier $\forall$

By the application of the first order language, we can formulate the various mathematical structures by simply adhering to their algorithm. The first order language thus constructs a significant part of the mathematical logic.

4. First Order Language For Groups, Rings and Vector Spaces

For the precise expression of the first order language, we will need the 'axioms of equality' that in standard notation used by one of the most prominent mathematician A.G. Hamilton in his book "LOGIC FOR MATHEMATICIANS" is given by (E7), (E8) and (E9). These axioms mentioned above forms the basis for further development in the abstract algebraic structures (4).

4.1.First order language for Groups:

For the sake of completion of our explanation, we write down the axiom of equality in the first step :

(E7)$A_1^2(x_1, x_2)$

(E8) $A_1^2(t_k, u) \rightarrow A_1^2\big(f_i^n(t_1, \dots, t_k, \dots, t_n)\big), f_i^n(t_1, \dots, u, \dots, t_n))$ where $t_1, \dots, t_k, \dots, t_n, u$ are any terms and f_i^n is any function letter of L.

(E9) $(A_1^2(t_k, u) \rightarrow (A_i^n(t_1, \dots, t_k, \dots, t_n) \rightarrow A_i^n(t_1, \dots, u, \dots, t_n)))$ where $t_1, \dots, t_k, \dots, t_n, u$ are any terms and A_i^n is any predicate symbol of L and in this case A_i^2 is interpreted as =.

These axioms (E7), (E8) and (E9) are called the axioms of equality.

The theory of groups is perhaps one of the most fundamental branches of pure mathematics which is best explicitly on a simple set of axioms, so let us use this concept to illustrate the properties of the composition of groups.

First, we must describe an appropriate first order language L, so let L_G be the first order language equipped with following alphabet of symbols:

- Variables $x_1, x_2, \ldots$
- The individual constant a_1 (identity)
- Function symbols f_1^1, f_1^2 (inverse, product)
- Predicate symbol =
- Punctuation (,)
- Logical symbols $\sim, \rightarrow$
- Quantifier $\forall$

Now we define G to be the extension K_{L_G} where the proper axioms are (E7), (E8) , (E9) and the following :

(G1) $f_1^2(f_1^2(x_1, x_2), x_3) = f_1^2(x_1, f_1^2(x_2, x_3))$ (associativity property)

(G2) $f_1^2(a_1, x_1) = x_1$ (left identity)

(G3) $f_1^2(f_1^1(x_1), x_1) = a_1$ (left inverse)

(G1), (G2), (G3) are the translations of the usual group axioms.

Thus, the first order language was designed in accordance with the subject matter we wish to study. First order language hence is considered a powerful instrument in the field of mathematical logic because it allows for analytically accurate and formal representation of the mathematical statements and their associated properties. Now in the similar way, we try to write or express the properties of rings and vector spaces. Here we will be using the same axioms of equality that are standard for every such idea of Abstract algebra.

4.2. First order language for Rings:

A ring is another significant section in the realm of abstract algebra. A ring is basically an algebraic structure armed with two fundamental binary operations – addition and multiplication.

In order for an algebraic structure to be considered a ring, it should be an Abelian group under addition and a semi-group under multiplication.

(R, +, .) is considered to be a ring if it follows the following properties.

(1) (R, +) is an abelian group.

(2) (R, .) is a semi group.

(3) It follows the distributive law:

(i) a (b+c) = a.b + a.c

(ii) (a+b)c = a.c + b.c

We must describe an appropriate first order language L, so let L_R be the first order language equipped with following alphabet of symbols:

- Variables $x_1, x_2, ...$
- The individual constant a_1 (identity)
- Function symbols $f_1^1, f_1^2, f_1^{\ 3}$ (inverse, product and sum respectively)
- Predicate symbol =
- Punctuation (,)
- Logical symbols $\sim, \rightarrow$
- Quantifier $\forall$

Now we define G to be the extension K_{L_R} where the proper axioms are (E7), (E8) , (E9) and the following:

(R1) $f_1^3(f_1^3(x_1, x_2), x_3) = f_1^3(x_1, f_1^3(x_2, x_3))$ (associative property in addition)

(R2) $f_1^3(a_1,\ x_1) = x_1$ (identity)

(R3) $f_1^{\ 3}(f_1^1(x_1), x_1) = a_1$ (inverse)

(R4) $f_1^{\ 3}(x_1, x_2) = f_1^{\ 3}(x_{2,} x_1)$ (commutative)

(R5) $f_1^2(f_1^2(x_1, x_2), x_3) = f_1^2(x_1, f_1^2(x_2, x_3))$ (associative property in multiplication)

(R6) $f_1^2\left(x_1, f_1^3(x_2, x_3)\right) = f_1{}^3(f_1^2(x_1, x_2), f_1^2(x_1, x_3))$ (left distribution)

(R7) $f_1^2(f_1^3(x_1, x_2), x_3) = f_1^3\ (f_1^2(x_1, x_3), f_1^2(x_2, x_3))$ (right distribution)

(R1), (R2), (R3), (R4), (R5), (R6) and (R7) are the translations of the usual ring axioms.

4.3. First order language for Vector Spaces:

Next our objective is to write the first order language in the case of vector spaces. Vector space is one of the most sought-after topics of algebra.

Let, (F, +, .) be a field. Then the elements of F will be called the scalars. Let, V be a non- empty set whose elements are called the vectors. Then V is a vector space over the field F if there is defined an internal composition in V called the addition vectors and is denoted by '+'. Following are the properties V need to adhere to in order to become a vector space:

(1) (V, +) is an Abelian group.

(2) V is closed with respect to the scalar multiplication.

Now, for any α, β $\in F$, a, b$\in V$, we have the following:

(3)(i) α (a + b) = α.a + α.b

(ii) (α + β)a = α,a + β.a

(iii) (α β)a = α (βa)

(iv) 1.a = a,   1 is unit element of F.

Here we must describe an appropriate first order language L, so let L_V be the first order language equipped with following alphabet of symbols:

- Variables $x_1, x_2, \ldots$
- The individual constant a_1 (identity)
- Function symbols $f_1^1, f_1^2, f_1^{\ 3}$ (inverse, product and sum respectively)
- Predicate symbol =
- Punctuation (,)
- Logical symbols $\sim, \rightarrow$
- Quantifier $\forall$

Now we define G to be the extension K_{L_V} where the proper axioms are (E7), (E8) , (E9) and the following:

(V1) $f_1^3(f_1^3(x_1, x_2), x_3) = f_1^3(x_1, f_1^3(x_2, x_3))$ (associative property in addition)

(V2) $f_1^3(a_1,\ x_1) = x_1$ (identity)

(V3) $f_1^{\ 3}(f_1^1(x_1), x_1) = a_1$ (inverse)

(V4) $f_1^{\ 3}(x_1, x_2) = f_1^{\ 3}\ (x_{2,} x_1)$ (commutative)

(V5) V is closed with respect to the scalar multiplication.

(V6) f_1^2 (α , $f_1^3(a, b)$) = f_1^3(f_1^2(α, a), f_1^2(α, b)) (Distributivity of scalar multiplication over vector addition)

(V7) f_1^2 ($f_1^3(\alpha, \beta)$, a) = f_1^3(f_1^2(α, a), f_1^2(β, a)) (Distributivity of vector multiplication over scalar addition)

(V8) ${f_1}^2({f_1}^2(\alpha, \beta)$, a) = ${f_1}^2(\alpha, {f_1}^2(\beta,\ a))$ (associativity)

(V9) ${f_1}^2(1, a) = a$ and 1 is a unit element of F.

Therefore, (V1), (V2), (V3), (V4), (V5), (V6), (V7), (V8) and (V9) are the translations of the usual vector space axioms.

5.Conclusion and future scope

Mathematical logic serves as the foundation for much of the modern-day mathematical study and its analysis, upon which the theory of Abstract algebra is developed and spread further. The formal system in mathematical analysis is a structured system of symbols and rules used in order to write down the precise and unambiguous mathematical statements. It equips us with the necessary tools for constructing and thoroughly analysing formal structures, as well as providing us with the validity of the mathematical arguments. So far, we have been acquainted with the algebraic structures such as groups and rings, as well as their logical foundations. In Abstract algebra, it offers a rigorous framework to establish the algebraic structures by the suitable application of the predicate symbols, functional symbols, individual constants, punctuations and quantifiers.

In our work so far, we have analysed the formal systems and the first order language for the complicated structures like rings and vector spaces. In the context of abstract algebra, especially as the field evolves, the formal languages will continue to play a predominant role in the further development and logical interpretation of the different algebraic structures. The axiom scheme along with the first order language comprises a part second to none. Equipped with the axiom schemes and logical deductive, it paves the way for the diverse logical interpretation. In the accurate formulation and composition of various abstract algebraic structures, the axiom schemes provide a systematic and generalized template for expressing the obligatory properties applied in these structures, while the first order language offers accurate formal language crucial to articulate them. It is the formulative basis for expressing mathematical statements, logical rules and reasoning precisely. As a result, the interplay of the axiom schemes along with the first order language is mandatory in the precise formalization and deeper understanding of abstract algebra. The logic moves past the conventional formal system by the thorough incorporation of cognitive perspectives into the structured logical theory (3). Apart from its wide variety of applications in the field of modern abstract algebra, it has also found utilization in the branches of philosophy, logic puzzles, reasoning, cryptography and computer science.

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