A STUDY ON CONFORMAL MAPPING AND ITS APPLICATIONS

Dissertation submitted to the Department of Mathematics in fulfilment of the requirement for the award of the degree of Master of Science

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CERTIFICATE

This is to clarify that **NUR SAMSUD SK** bearing **Roll No MAT-24/23** and **Regd. No. MSSV-0023-101-001443** has prepared his dissertation entitled "**CONFORMAL MAPPING AND ITS APPLICATIONS"** submitted to the Department of Mathematics, **MAHAPURUSHA SRIMANTA SANKARDEVA VISWAVIDYALAYA,** Nagaon, for fulfilment of MSc. degree, under guidance of me and neither the dissertation nor any part thereof has submitted to this or any other university for a research degree or diploma.

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DECLARATION

I, **Nur Samsud Sk** , bearing ROLL No- **MAT-24/23** student of final semester, Department of Mathematics , **Mahapurusha Srimanta Sankaradeva Vishwavidyalaya** (MSSV), do hereby declare that the work incorporated in this dissertation entitled "**Conformal Mapping and Its Applicatons**", for the award of degree of Master of Science in Mathematics, has been carried out and interpreted by me under the supervision of **DR. MIRA DAS** , Assistant Professor, Department of Mathematics , (MSSV) Assam. This dissertation is original and has not been submitted by me for the award of degree of diploma to any other University or Institute. I have faithfully and accurately cited all my sources, including books, journals, handouts and unpublished manuscripts, as well as any other media, such as the internet, letters or significant personal communication.

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CONTENTS

ABSTRACT

The calculation of conformal mapping for irregular domains is a crucial step in deriving analytical and semi-analytical solutions for irregularly shaped tunnels in rock masses using complex theory. The optimization methods, iteration methods, and the extended Melentiev's method have been developed and adopted to calculate the conformal mapping function in tunnel engineering. According to the strict definition and theorems of conformal mapping, it is proven that these three methods only map boundaries and do not guarantee the mapping's conformal properties due to inherent limitations. Notably, there are other challenges in applying conformal mapping to tunnel engineering. To tackle these issues, a practical procedure is proposed for the conformal mapping of common tunnels in rock masses. The procedure is based on the extended SC transformation formulas and corresponding numerical methods. The discretization codes for polygonal, multi-arc, smooth curve, and mixed boundaries are programmed and embedded into the procedure, catering to both simply and multiply connected domains. Six cases of conformal mapping for typical tunnel cross sections, including rectangular tunnels, multi-arc tunnels, horseshoe-shaped tunnels, and symmetric and asymmetric multiple tunnels at depth, are performed and illustrated. Furthermore, this article also illustrates the use of the conformal mapping method for shallow tunnels, which aligns with the symmetry principle of conformal mapping. Finally, the discussion highlights the use of an explicit power function as an approximation method for symmetric tunnels, outlining its key points.

Keywords: Conformal mapping; Tunnel engineering; Complex analysis; Schwarz–Christoffel mapping; Existence and uniqueness theorem; Boundary correspondence theorem.

REVIEW OF LITERATURE

The term "complex analysis" refers to the calculus of complex-valued functions f(z) depending on a single complex variable z. Complex analysis is the culmination of a deep and far-ranging study of the fundamental notions of complex differentiation and integration, and has an elegance and beauty. Conformal mapping is transformation in complex analysis that preserves angles locally. In complex analysis, conformal mappings are typically associated with analytic function.

Several researchers have studied in this area. Before going to prepare this dissertations we have studied about some research of some reserchers. Reserchers are exploring how conformal mapping can be used to design and optimize the shapes of objects.

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CHAPTER-1

Basic Fundamental concepts and Basic Facts

1.1 Introduction

In mathematics a conformal mapping is function which preserve angle. In 1569 the Flemish cartographer Geradus Mercator devised a cylindrical map projection known to ancient Greek is the sterographic projection and both example are conformal. Conformal map is very important in complex analysis as well as physics and engineering.

Before we start conformal mapping we need to understand about Mapping. In linear it is a mathematical relation such that each element of a given set is associated with an element of another set. A complex function $w = f(x)$ can be regarded as a mapping or transformation of the points in the $z = x + iy$ plane to the points of the $w = u + iv$ plane.

If the transformation $U = u(x, y)$ and $V = v(x, y)$ the point (x_0, y_0) of $x\,y$-plane is mapped into the point (u_0, v_0) of $U\,V$-plane the intersecting curve

$c_1$and c_2 are respectively mapped into the curve c_1' and c_2' and they intersect at (u_0, v_0).

If the transformation is such that the angle at (x_0, y_0) between c_1 and c_2 is equal to the angle at (u_0, v_0) between c_1' and c_2' is equal in magnitude as well as sense then the transformation is called conformal mapping or conformal transformation. In conformal mapping the magnitude or angle is same and the direction of angle is also same.

Note: (i) The scaling factor and the rotation angle for a given transformation $f(z)$ depend only on Z and not on curves passing through Z.

(ii) Conformality depends on the given point. It is different at various points in the plane.

Any conformal map from an open subset of Euclidean space into the same Euclidean space of dimension three or greater can be composed from three types of transformations a homeothely,an isometry and a special conformal transformation. Applications of conformal mapping exist in aerospace engineering, in biomedical sciences, in applied math, in earth sciences and in electronics. In cartography several named map projections, including the mercator projection and stereographic projection are conformal. They are specially useful for use in marine nevigation because of its unique property of representing any course of constant bearing as a straight segment. Such a course known as a rhumb (or mathematically a loxodrome) is preferred in marine navigation because ships can sail in a constant compass direction. Coqnformal mappings are invaluable for solving problems in engineering and physics that can be expressed in terms of functions of a complex

variable yet exhibit inconvenient geometries. By choosing an appropriate mapping, the analyst can transform the inconvenient geometry into a much more convenient one. For this reason any function which is defined by a potential can be expressed in terms of functions of complex variable yet exhibit inconvenient geometries. By choosing an appropriate mapping the analyst can transform the inconvenient geometry into a much more convenient one. For this reason any function which is defined by a potential can be transformed by a conformal map and still remain governed by a potential. Example- In physics of equations defined by a potential include the electromagnetic field, the gravitational field and in fluid dynamics, potential flow,which is an approximation to fluid flow assuming constant density, zero viscosity and irrotational flow. One example of a fluid dynamics application of a conformal maps are also valuable in solving non-linear partial differential equations in some specific geometries. Such analytic solutions provide a useful check on the accuracy of numerical simulations of the governing equation.

A non conformal projection can be used in a limited domain such that the projection is locally conformal. Glueing many maps together restores roundness. To make a new sheet from many maps or to change the center, the body must be projected seamless online maps can be very large mercator projections so that any place can become the maps centre then the map remains conformal. However it is difficult to compare length or areas of two far off figures using such a projection. The universal transverse Mercator co-ordinate system and the Lambert system in France are projections that support the trade off between seamlessness and scale variability. Maps reflecting directions such as a nautical chart or an aeronautical chart are projected by conformal projections.

1.2 BASIC DEFINTIONS

A region Ω is said to be simply connected if every closed curve in Ω is homotopic to a point. Let D be a domain and let u(z) be a continuous function from D to the extended real line $[-\infty, \infty)$. We say that u(z) is subharmonic if for each $z_0 \in D$ there is $\epsilon > 0$ such that u(z) satisfies the mean value inequality this inequality is automatically satisfied. If f is defined in a set S and if for each Z in S there is only one value f(z) of z , then f is said to be single valued. Let f be a function defined in a set S and Let z_0 be a limit point of S then A is said to be a limit of f(z) at z_0 if for any $\epsilon > 0$ there exist a $\delta > 0$ such that $|\ f(z) - A| < \epsilon$ for all z in S other than z_0 with $|\ z - z_0\ | < \delta$. δ is a function of ϵ say (ϵ). A function which has no singularities in a region other than a finite number of poles is said to be meromorphic in that region. A function (z) is analytic at a point z_0 if it has a complex derivative $f'(z_0)$ at z_0 and in some neighborhood around z_0. A region which is not simply connected is called multiply connected region. More precisely, Ω is said to have the finite connectivity n if the complement of Ω has exactly n components and infinite connectivity if the complement

has infinitely many components. A mapping f: $S \rightarrow S'$ is said to be one to one if (x) = (y) only for x = y; it is said to be onto if (S) = S'. A mapping with both these properties has an

In this situation, if f and f^{-1} are both continuous we say that f is a topological mapping or a homeomorphism. A linear fractional transformation can be defined as a transformation. Here, az + b and cz + d are linear. Also a, b, c, d,and z are complex numbers. This transformation can be associated with a matrix given by $(a\ b\ c\ d)$plane is expressed in parametric form x = (t); y = (t) where a ≤ t ≤ b and (t), $(t)$are continuous function. A transformation w = (z), defined on a domain D, is referred to as a conformal mapping . When it is conformal at each point in D. That is, the mapping is conformal in D if f is analytic in D and its derivative f has no zeros there.

CHAPTER 2

Types of Conformal Mapping

2.1 Introduction

Conformal mapping in essence preserves angles during a transformation. It's a conformal primarily used in complex analysis where it maps points in one complex plane to another while maintaining the angles between curves. These are various types of conformal mappings including those based on complex functions(like Mobius transformations) and those involving transformations in higher dimensions. These transformations can be categorized by their properties such as whether they preserve orientation or just the magnitude of angles.

2.1.1 Homothetic Transformation

A homothety (or dilation) is just an expansion or contraction, in which a figure is 'zoomed' (in/out) from a center. For instance, figure 1 shows the effect of a homothety centred at O. In the left panel, the triangle ABC is magnified by a factor of 2 to become triangle A0B0C 0 . We say that 4A0B0C 0 is the image of 4ABC under a homothety centred at O with scale factor 2. Mathematically we have OA0 = 2OA, OB0 = 2OB etc. In contrast, figure 2 shows the effect of a homothety centred at O with scale factor −2, i.e. OP0/OP = −2 for each point P.

The minus sign indicates that P 0 is outside the segment OP for any point P (e.g. A0 is outside the segment OA).

More formally, a homothety h is a transformation on the plane defined by a center O and a real number k, which sends any point P to another point P 0 = h(P), such that OP0 OP = k. The number k is the scale factor. Note that the transformation is meaningful for any point other than O itself; we either leave h(O)

as undefined, or set h(O) = O as a convention. The homothety centred at with scale factor k is sometimes denoted by H(O, k). We emphasize again on the fact that this k can be negative, indicating that the image P 0 lies on the 'other side' of O w.r.t. point P. Here is an applet where you can use the slider to change the scale factor k and see the effect of the homothety:

Following are some properties of Homothety:

1. The image of any object under a homothety is similar to the original object.
2. A homothety is defined uniquely by where any 2 points are mapped to.
3. The slope of a line is preserved under homothety. Hence, parallel lines are preserved.
4. Angles are preserved, meaning that ∠ABC = ∠A0B0C 0.
5. H(O, −1) is a reflection through the center, and is also a 180◦ rotation about the center.

6. For k 6= 1, the set of lines that remain invariant under H(O, k) are pecisely the lines that pass through the center O.

2.1.2 Isometric Transformation

A shape-preserving transformation in the plane or space is an isometric transformation. As objects move in the coordinate plane, they transform in geometric terms. In other words, a transformation transforms a set of coordinate points into a different set of coordinates. A transformation alters a figure's scale, shape, or location and generates a new image. An isometry does not alter the size or shape of the figure. The size or shape of the figure is not altered by an isometry. A transformation in geometry is either rigid or non-rigid. Isometry is another term for a rigid transformation.

Thus, by definition, a transformation maps or transfers an initial image (preimage) to a final image (image). An object's size, location, or orientation may all change during transformation.

A transformation that retains the distance and angles between the original shape and the shape that has been transformed is an isometric transformation. Any image in a plane can be transformed using various operations.

2.2 Categories of Transformation

A transformation can be classified into one of two categories.

Rigid or Isomeric Transformation-

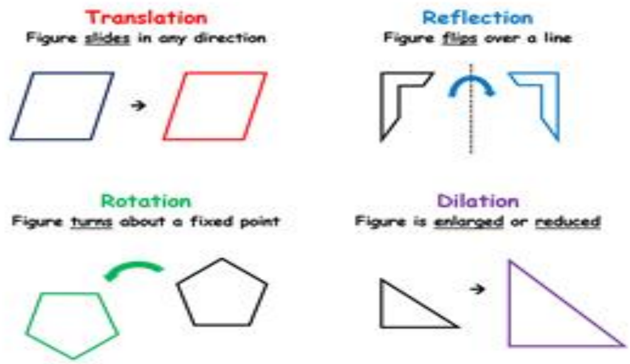
A transformation in which the preimage's scale remains constant is called a rigid transformation.

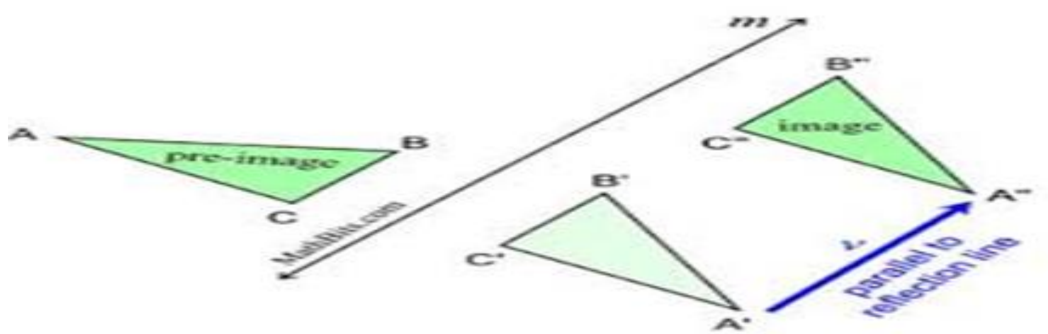
Non-Rigid or Non-Isomeric Transformation-

A transformation in which the preimage's scale changes but not the shape is called a non-rigid transformation.

There are five different forms of transformations depending on how the image is modified. One of them is a non-rigid transformation, while the others are all rigid transformations. The five transformations are as follows:

- Translation
- Reflection
- Rotation
- Glide Reflection
- Enlargement



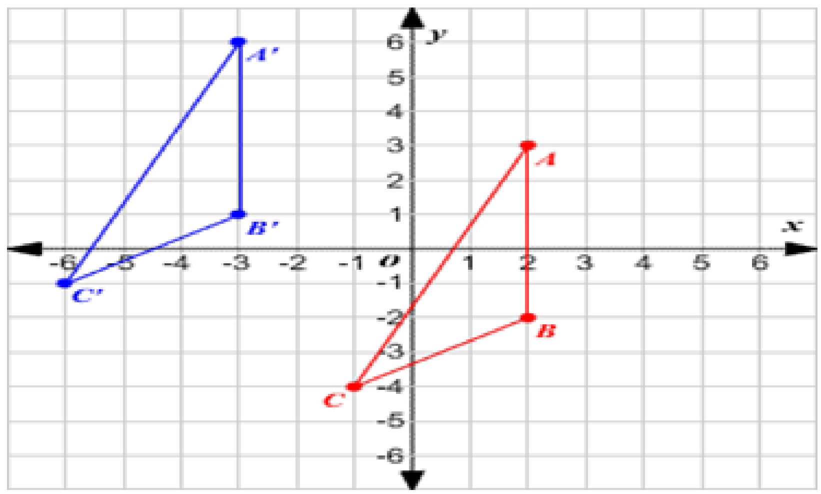


2.2.1 Translation (Slide)

You slide it around when you transform an object into a plane. An isometry is a translation in the plane that shifts a fixed distance in a fixed direction at each point in the plane. You're not going to flip, turn, twist, or bop it. Actually, for translations, you know where they all go if you know where one point goes. Translations retain orientation: left remains left, right remains right, top remains top, and the bottom remains bottom. Proper isometries are referred to as isometries that cover orientations. Sliding a figure is a translation.

A translation is described by stating the movement's path and the object's distance travelled.

For example, in the figure below triangle ABC is translated 5 units to the left and 3 units up to get the triangle A'B'C' image.



Formula to Calculate Translation

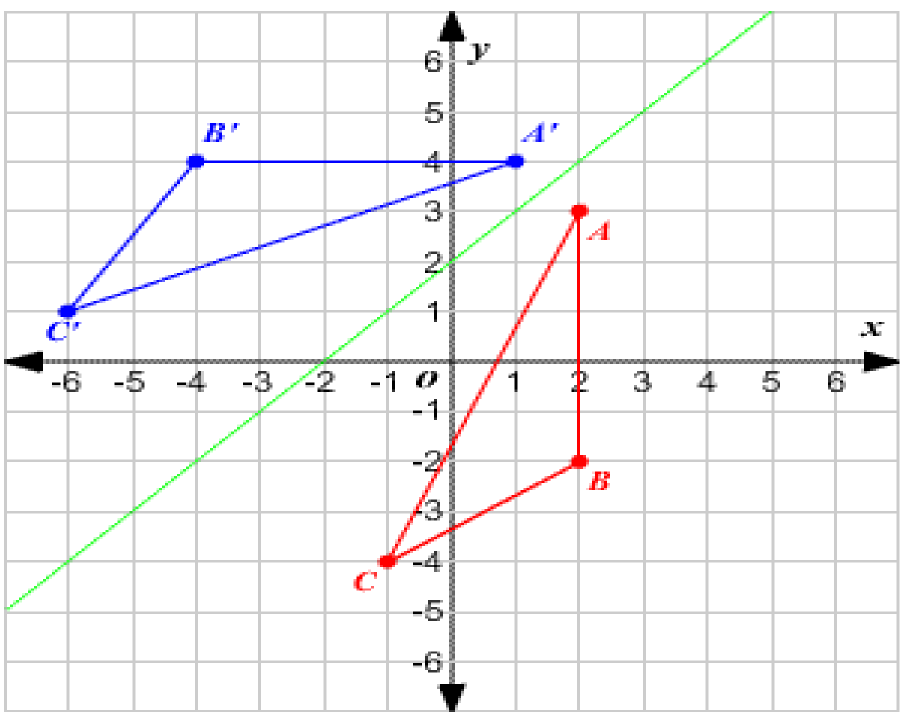
We can draw the translation in the coordinate plane if we know the direction and how far the figure should be moved. Using P'(x+a, y+b) to translate the point P (x, y), a unit right and b units up.

2.2.2 Reflection

Reflection is the second type of transformation. A reflection in the plane shifts an object into a new location that is the original position's mirror image. The mirror, called the axis of reflection, is a line. If you know the axis of reflection, everything there is to know about isometry is understood. The orientation changes in the reflection.

Reflections are tricky since the reference frame shifts. Depending on the axis of reflection, the left can become right, and the top can become lower. The image and object lie on opposite sides of the mirror line. The mirror line is perpendicular to the line connecting any object point to its corresponding image point. The mirror line divides the plane into two equal sections that overlap and is called an axis of bilateral symmetry.

They are called improper isometries since reflections alter the orientation. For example, the following figure shows the ABC triangle mirrored across the y=x+2 axis.



Formula to Calculate Reflection

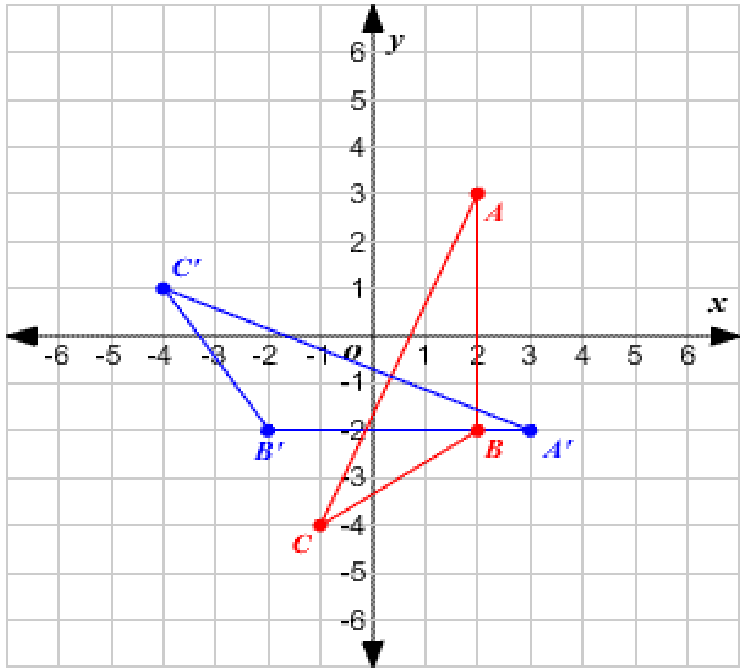
The law for x-axis reflection is point P(x, y) becomes P'(x, -y)

2.2.3 Rotations

The rotation is a third type of transformation. A rotation is a geometric transformation that involves turning or rotating an object around a fixed point called the center of rotation. The angle of rotation determines the size of the turn. A rotation requires an isometry that holds one point fixed and changes a certain angle relative to the fixed point to all other points. You have to know the pivot point, called the center of the rotation, in order to define a rotation. You also have to know the amount of rotation.

An angle and a direction determine this. For example, you might rotate a figure at an angle of 90^0 around point P, but you need to know whether the rotation is clockwise or counterclockwise. Some examples of rotations concerning some points are shown in the figure below.

Other than the rotation of the identity, rotations have one fixed point: the rotation center. You don't change it if you turn a point around, so it doesn't have any scale. A rotation retains orientation, as well. All rotate in the same direction at the same angle, so left remains left, and right remains right. The figure below shows a 90 ° clockwise rotation of the triangle ABC around the origin.



Formula to Calculate Rotation

Phase 1: In the specified figure, pick any point and link the chosen point to the rotation center.

Phase 2: Locate and attach the image of the chosen point to the center of the rotation.

Phase 3: Calculate the angle of the two lines between them. The angle sign is dependent on the direction of rotation.

2.2.4 Glide Reflections

A glide reflection is a symmetry operation in 2-dimensional geometry that combines reflection over a line and translation along that line into a single operation. Note that there has been a shift in orientation. The two-component transformations can be performed in any order in a glide reflection. The footprints are glide reflections of each other. The initial "move" on the left foot is seen on the left, followed by the "step" on the right foot, which is the "result" of the glide reflection on the left. Glide reflections are improper isometries because the orientation has shifted.

The translation and the axis of reflection are parallel. When you know how two points are moved, it is simple to decide the axis of reflection.

2.2.5 Enlargement

Dilation, also known as enlargement, is a transformation in which the size of an object changes but not its shape. The scale factor of the enlargement determines the size change. Enlargement is also known as a scale transformation.

When explaining an enlargement (or dilation), we must always provide the following information:

(i) The location of the enlargement's base, which indicates where the enlargement is measured.

(ii) The enlargement's scale factor, which indicates how much the object has been expanded or reduced to create the picture.

Enlargement and reduction are the two main forms of dilations. The image is larger than the object (enlargement) when the scale factor (k) is greater than one, and the image is smaller than the object when the scale factor (k) is less than one (reduction).

The scale factor of dilation may be positive or negative. The object and image are on the same side of the center when the scale factor is positive. The object and image are on opposite sides of the center when the scale factor is negative.

2.3 **Special Conformal Transformation**

In projective geometry, a **special conformal transformation** is a linear fractional transformation that is *not* an affine transformation. Thus the generation of a special conformal transformation involves use of multiplicative inversion, which is the generator of linear fractional transformations that is not affine.

In mathematical physics, certain conformal maps known as spherical wave transformations are special conformal transformations.

2.3.1 Vector presentation

A special conformal transformation can be written as

It is a composition of an inversion ($x^\mu \rightarrow x^\mu/\mathrm{x}^2 = y^\mu$), a translation ($y^\mu \rightarrow y^\mu - b^\mu = z^\mu$), and another inversion ($z^\mu \rightarrow z^\mu/\mathrm{z}^2 = x'^\mu$)

Its infinitesimal generator is

Special conformal transformations have been used to study the force field of an electric charge in hyperbolic motion.

2.3.2 Projective presentation

The inversion can also be taken to be multiplicative inversion of biquaternions *B*. The complex algebra *B* can be extended to P(*B*) through the projective line over a ring. Homographies on P(*B*) include translations:

The homography group G(B) includes of translations at infinity with respect to the embedding $q \to \mathrm{U}(q:1)$;

The matrix describes the action of a special conformal transformation.

2.4 Group Property

The translations form a subgroup of the linear fractional group acting on a projective line. There are two embeddings into the projective line of homogeneous coordinates: $z \to [z:1]$ and $z \to [1:z]$. An addition operation corresponds to a translation in the first embedding. The translations to the second embedding are special conformal transformations, forming translations at infinity. Addition by these transformations reciprocates the terms before addition, then returns the result by another reciprocation. This operation is called the parallel operation. In the case of the complex plane the parallel operator forms an addition operation in an alternative field using infinity but excluding zero. The translations at infinity thus form another subgroup of the homography group on the projective line.

CHAPTER-3

Mobius Transformation

3.1 Introduction

A Mobius transformation or Mobius inversion is a geometric transformation that takes a line segment and swivels it through 180 degrees and then connects the resulting two endpoints together. Imagine an equilateral triangle with a line segment that is perpendicular to one side of the triangle, this line segment is our starting point. Our goal is to invert the triangle after transforming our line segment into another part of the triangle. The process to accomplish this goal is as follows---

We start by rotating the line segment about its midpoint through 180 degrees, we now have three segments connected by two points. The next step is the most complicated one and it involves the inversion. In this step we connect the midpoints of the two segments that result from our first rotation.

When we connect these two midpoints we get a figure that looks like an infinity symbol, or as it is called a Mobius strip.

The image below shows this process visually. Every point on this Mobius strip can be thought of as being a well-defined point in space, some points are exceptional however, for example points A and B on our graph would be considered exceptional points since no line segment can be drawn to point A and B without crossing over itself (self intersection) or passing through another point on the curve.

3.2 Mobius Group

The Mobius transformations are special because if we do one and then reverse the operation we get the same result as if we did it in the opposite order. Also, doing a sequence of these transformations will result in a series of equivalent transformations.

The group of Mobius transformations is denoted by M and has 8 elements that can be combined in pairs to produce an equivalent transformation; this is important to note because a Mobius transformed figure can have up to 6 points that are considered exceptional points.

How is Mobius Transformation related to other concepts in Mathematics?

If we look at our Mobius strip as a piece of paper, it would have the shape of a normal bezier curve which is another concept in mathematics. Bezier curves are artifacts that arise from Mobius transformations in 3D. There is one important difference between the Mobius transformation and Bezier curve however, Beziers only deform shapes on 2D surfaces, whereas the Mobius strips can deform 3D shapes such as spheres. The concept behind this fact is that points on an object can be thought of as being localized on an extended surface. In reality, each point is defined by two perpendicular line segments and two perpendicular planes intersecting between these line segments at right angles.

Significance of Mobius Transformation:

Mobius transformations are important for many reasons, the one that is probably most significant is that they are a link between Euclidean space and 3-manifolds. A Mobius strip can be deformed into a 3-manifold without distorting its surface area or volume.

Mobius transformations are also important in Electromagnetism because they give rise to the concept of electromagnetic field tensors and flux which only arises from an object being twisted in a Mobius fashion.

If two 2D surfaces, such as the inside and outside of a sphere, are twisted in such a way that they no longer touch each other when inverted (in Mobius fashion), they are said to be Mobius invariant.

Mobius transformations are also used to determine the best placement of detectors on a spacecraft to minimize self-intersection and maximize overlap between detection zones.

A transformation is a transformation of the form

$$M(z) = \frac{az+b}{cz+d}$$

Where $a, b, c, d \in \mathbb{C}$ and with $ad - bc \neq 0$. Furthermore, M is a map $M: \mathbb{C} \to \mathbb{C}$. This map M can be extended to $M: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by defining $M\left(-\frac{d}{c}\right) = \infty$ and $(\infty) = \frac{a}{c}$.

Theorem: A Mobius transformations are conformal maps.

Before we can prove this theorem, we need the following lemma

Lemma: A Mobius transformation is the composition of a translation, inversion, reflection with rotational condition.

In this lemma we see the dilation. To understand this term, we use the following definition.

Definition: A dilation is a map $f: R^n \to R^n$ of the form $f(x) = \delta + \xi(x - \delta)$ where ξ is a non zero scalar and δ is a fixed point.

Now we know this, we can proof the lemma

$$M_1(z) = z + \frac{d}{c}$$

$$M_2(z) = \frac{1}{z}$$

$$M_3(z) = -\frac{(ad - bc)}{c^2} z$$

$$M_4(z) = z + \frac{a}{c}$$

Now an easy computation shows us that indeed $M_4 \circ M_3 \circ M_2 \circ M_1(z) = \frac{az+b}{cz+d}$

Now we can prove the Mobius transformation is conformal.

Proof: To prove the theorem it is enough to show that each M_i with $i = 1,2,3,4$ is conformal, since the composition of conformal maps is again conformal.

M_1 and M_2 are conformal because a translation is a conformal map, since the angle between two curves doesn't change when these curves are translated. M_2 is conformal since both inversion and reflection are conformal. That inversion is conformal is discussed in the previous section. Also reflection is conformal, since it doesn't change the angle between two curves. Finally M_3 is conformal. This is true because dilation is nothing more than scalar multiplication and this doesn't change the angle between to curves. Also rotation doesn't change the angle, so also M_3 is conformal. This means that also the composition of the M_i is conformal and thus $M(z)$ is conformal.

3.3 Anti-Homographies

In this section we first will give the definition of an anti-homography. After that, we will look at the conformality of the anti-homographies. An anti-homographies is a transformation that looks like a Mobius transformation only with $\bar{z}$ and z. So an anti-homography $w: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is defined as $W(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$

Since an inversion in the unit sphere in the complex caseis given by $w = \frac{1}{\bar{z}}$, inversion is included in the set of anti-homographies. The most important thing we can say about anti-homographies.

3.3.1 Anti-Holomorphic Function

In this section we first give the definnitions of a holomorphic function and a anti holomorphic function. After that will look at the conformality of these functions. A holomorphic function is a complex-valued function that is differentiable in every point of $\mathbb{C}$. An anti-holomorphic function z is a function that is differentiable with respect to the complex conjugate $\bar{z}$. We also can define holomorphic and anti-holomorphic in terms of the Cauchy-Riemann equations. For a function $f = u + iv$, where $u = u(x, y)$ and $v = v(x, y)$ are real valued functions, we can say that

1. f is holomorphic iff f satisfies $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.
2. f is anti-holomorphic iff f satisfies $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.

   This last property can be used to check if a function f is conformal. Therefore, we have the following theorem

Theorem: Take $\boldsymbol{f}: \mathbb{R}^{\mathbf{2}} \to \mathbb{R}^2$ a function of class C^1 with a nonvanishing Jacobian. Then f as a map is conformal iff f as a function of $z \in \mathbb{C}$ is holomorphic or anti-holomorphic.

3.4 General Mobius Transformations

3.4.1 Mobius in $\mathbb{R}^2$

We have already defined Mobius transformation and we have seen that transformations are conformal. In this subsection we will see the Mobius transformation in detail First we have the following lemma

Lemma : A Mobius transformation $M(Z) = \frac{az+b}{cz+d}$ in $\mathbb{R}^2$ is the composition of inversions in spheres.

Proof: We distinguish two cases, the case $c = 0$ and the case $c \neq 0$.

First,if $c = 0$ we can say that M is an extended linear function and therefore it is a composition of inversions in spheres.

Now assume $c \neq 0$. Then we can write for $z \in \hat{\mathbb{C}} \setminus \left\{-\frac{d}{c}\right\}$ that

$$M(z) = \frac{a(cz+d) - ad + bc}{c(cz+d)}$$

$$= -\left(\frac{ad - bc}{c}\right) \cdot \left(\frac{1}{cz+d}\right) + \frac{a}{c}$$

So we can write M as the composition $t_3 \circ t_2 \circ t_1$ where t_2 is the extended reciprocal function, and thus a composition of inversions. Furthermore, t_1 and t_3 are the extended linear functions given by

$$t_1(z) = \begin{cases} cz + d \ \ if \ z \neq \infty \\ \infty \ \ if \ \ z = \infty \end{cases}$$

$$t_2(z) = \begin{cases} -\left(\frac{ad - bc}{c}\right) z + \frac{a}{c} \ \ if \ z \neq \infty \\ \infty \ \ if \ z = \infty \end{cases}$$

Also the extended linear functions are a composition of inversions and therefore,since both t_1 and t_2 as well as t_3 are compositions of inversions, it must hold that $M(z)$ is a composition of inversions as well which we wanted to prove.

Definition: An extended linear function is a function of the form $t(z) = az + b$ where $z, a, b \in \hat{\mathbb{C}}$ and $a \neq 0$. The extended linear function can be decomposed into $t = t_2 \circ t_1$ where (i) $t_1(z) = |a|z$ is a scaling

(ii) $t_2(z) = \frac{a}{|a|} + b$ is an isometry

Definition: The extended reciprocal function is a function t given by

$t(z) = \frac{1}{z}$ where $z \in \hat{\mathbb{C}} \setminus \{0\}$

The extended reciprocal function can be decomposed into $t = t_2 \circ t_1$ where

(i) $t_1(z) = \frac{1}{z}$ is an inversion

(ii) $t_2(z) = \bar{z}$ is a conjugate

Definition: For a Mobius transformation $M(z) = \frac{az+b}{cz+d}$ the matrix A given by

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix associated with $M(z)$.

To compute the composition of M_1 and M_2 with associated matrices A_1 and A_2 we can just compute the product $A_1 A_2$. This matrix product is now the associated matrix of the composition of M_1 and M_2.

Since this is not completely trivial, we will show that this results is true. Take two Mobius transformation $M_1 = \frac{a_1 z + b_1}{c_1 z + d_1}$ and $M_2 = \frac{a_2 z + b_2}{c_2 z + d_2}$ with associated matrices A_1 and A_2 respectively, where A_1 and A_2 are given by

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

Then with an easy computation we can see that the composition of M_1 and M_2 is given by $M_2 \circ M_1(z) = M_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right)$

$$= \frac{(a_2 a_1 + b_2 c_1)z + (a_2 b_1 + b_2 d_1)}{(c_2 a_1 + d_2 c_1)z + (c_2 b_1 + d_2 d_1)}$$

Which has associated matrix $A = \begin{pmatrix} a_2 a_1 + b_2 c_1 & a_2 b_1 + b_2 d_1 \\ c_2 a_1 + d_2 c_1 & c_2 b_1 + d_2 d_1 \end{pmatrix}$

With another computation it follows easily that $A_2 A_1 = A$, so indeed to compute the composition of Mobius transformations it is enough to compute.

CHAPTER 4:

Applications of Conformal Mapping

4.1 Introduction

Conformal mapping can be effectively used for constructing solutions to the Laplace equation on complicated planner domains that appear in a wide range of physical problems, including fluid mechanics, aerodynamics, thermo mechanics, electrostatic and elasticity. In this chapter, we will develop the basic techniques and theorems of complex analysis that impinge on the solution to boundary value problems associated with the planner Laplace equations and will expose three applications of conformal mappings that Saf-Sni, Rud, Ahl.

Application : Fluid Flow

Conformal mapping techniques can be used in a clever way to analyze problems in fluid flow mechanics in two-dimentional domains, when the flow is incompressible, irrotational and steady. Two-dimensional fluid mechanics problems are relevant when the fluid is thin in the third dimension, in which case the fluid velocity is often negligible in that direction or otherwise uniform along that direction.

4.2 Boundary value problems and fluid flow

A boundary value problem is a differential equation with a set of additional constraints called boundary conditions. A solution of such boundary value problem satisfies the differential equation and boundary conditions. Boundary value problems arise from several fields in physics as in wave equations, potential forces and steady temperatures. Among the earliest boundary valued problems studied is the Dirichlet problem of finding harmonic functions. There are several types of boundary conditions. A boundary condition which specifies the value of the function itself at the boundary is a Dirichlet boundary condition, while a boundary and not the value of the function itself is a Neumann boundary condition. When the boundary has the form of a curve or surface that gives a value to the normal derivative and the variable itself then this condition is a Cauchy boundary condition.

Consider a planner steady state fluid flow, with velocity vector field: $\vartheta(x, y) = \big(u(x, y), v(x, y)\big)^T$

Where (x, y) represents the domain occupied by the liquid, while the vector $\vartheta(x, y)$ represents the instantaneous velocity of the fluid at (x, y). Recall that the flow is incompressible if and only if it has vanishing divergence, that is

$$\nabla\vartheta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Incompressibility means that the fluid volume does not change as it flows. Most liquids, including water are for all practical purposes, incompressible. On the other hand, the flow is irrotational if and only if it has vanishing curl, that is

$$\nabla \times \vartheta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Irrotational flows have no vorticity and hence no circulation. A flow that is both incompressible and irrotational is known as an ideal fluid flow. In many physical regimes, liquids(and, although less often, gases) behave as ideal fluids. The velocity vector field $\vartheta(x, y)$ includes an ideal fluid flow if and only if $f(z) = u(x, y) - iv(x, y)$ is analytic. The corresponding complex function f is called the complex velocity of the fluid flow. Under the flow induced by the velocity vector field $\vartheta = \big(u(x, y), v(x, y)\big)^T$, the fluid particles follow the trajectories $z(t) = x(t) + iy(t)$ obtained by integrating the system of ordinary differential equations:$\frac{dx}{dt} = u(x, y)$ and $\frac{dy}{dt} = v(x, y)$.

Now, suppose that the complex velocity $f(z)$ admits a complex primitive $X(z) = \emptyset(x, y) + i\varphi(x, y)$

Then, $\frac{dx}{dz} = \frac{\partial \emptyset}{\partial x} - i\frac{\partial \emptyset}{\partial y} = u - iv$

Thus, $\nabla \emptyset = \vartheta$ and hence the real part $\emptyset(x, y)$ of the complex function $X(z)$ defines a velocity potential for the fluid flow. For this reason, the anti-derivative $X(z)$ is known as a complex potential function for the given fluid velocity field. Since the complex potential is analytic, its real aprt, the potential function is harmonic and therefore satisfies the Laplace equation $\Delta \emptyset = 0$. Conversely, any harmonic function can be viewed as the potential function for some fluid flow. The real fluid velocity is its gradient : $\vartheta = \nabla \emptyset$ and is incompressible and irrotational. The harmonic conjugate $\varphi(x, y)$ to the velocity potential also plays an important role and in fluid mechanics is known as the stream function. It also satisfies the Laplace equation $\Delta \varphi = 0$ and the potential and stream function are related by the Cauchy-Riemann equations:

$\frac{\partial \emptyset}{\partial x} = u = \frac{\partial \varphi}{\partial y}$ and $\frac{\partial \emptyset}{\partial y} = v = -\frac{\partial \varphi}{\partial x}$

Example : Consider a complex potential function $X(z) = z = x + iy$, the velocity potential is $\emptyset(x, y) = x$ and the steam function is $\varphi(x, y) = y$. The complex derivative of the potential is the complex velocity $f(z) = \frac{dX}{dz} = 1$

Solution: Let $\theta(\varepsilon) = \phi(\xi, \eta) + i\psi(\xi, \eta)$ be an analytic function representing the complex potential function for a steady state fluid flow in a planer domain $\zeta \in D$. Composing the complex potential $\theta(\zeta)$ with a one to one conformal map $\zeta = g(z)$

leads to a transformed complex potential $X(z) = \theta\big(g(z)\big) = \phi(x, y) + i\psi(x, y)$ on the corresponding domain $\Omega = g^{-1}(D)$, thus we can use conformal maps to construct fluid flows in complicated domains from known flows in much simpler domains. We will restrict our study to planner fluid flows around a closed and bounded subset $D \subset \mathbb{R}^2 = \mathbb{C}$, representing a cross-section of a cylindrical object

Figure: Cross section of cylindrical object [Olv]

The (complex) velocity and potential are defined on the complementary domain $\Omega = \mathbb{C} \setminus D$ occupied by the fluid. The velocity potential $\phi(x, y)$ will satisfy the Laplace equation $\Delta\phi = 0$ in the exterior domain Ω. For a solid object, we should impose by homogeneous Neumann boundary conditions $\frac{\partial \phi}{\partial \eta} = 0$

On the boundary $\partial\Omega = \partial D$ indiacating that there is no fluid flux into the object. We note that, since it preserves angles and hence the normal direction to the boundary, a conformal map will automatically preserve the Neumann boundary conditions. We shall assume our object is placed in a uniform horizontal flow e.g., a wind tunnel, as sketched in figure.

Thus, far away the object will not affect the flow and so the velocity should approximate the uniform velocity field $\vartheta = (1,0)^T$, where for simplicity we choose our physical units so that the fluid moves from left to right with an asymptotic speed equal to 1. Equivalently, the velocity potential should satisfy:

$\phi(x, y) \approx x$ so $\nabla\phi \approx (1,0)$ when $x^2 + y^2 \ggg 0$.

An alternative physical interpretation is that we are located on an object that is moving horizonally to the left at unit speed through a fluid that is initially at rest. Think of an airplane flying through the air at constant speed. If we adopt a moving co-ordinate system by sitting inside the airplane, then the effect is as if the plane is sitting still while the air is moving towards us at unit speed.

4.3 Horizonal plate

A simple example is the flat plate moving horizonally through the fluid. The plate's cross section is a horizontal segment and for convenience we may consider this segment to be $D = [-1,1]$ located on the real axis. If the plate is thin and smooth, it will affect the horizontal flow and in this case the velocity potential is given by $\phi(x, y) = x$ for $x + iy \in \Omega = \mathbb{C} \setminus [-1,1]$

4.4 Circular disk

We have seen previously that the Joukowski conformal map defined by $\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ squashes the unit circle $|z| = 1$ into the real line segment $[-1,1]$in the ζ-plane and so $\varsigma = g(z)$ will map the fluid flow outside the unit disk to the fluid flow past the line segment. The complex potential for horizontal plate flow was $\theta(\varsigma) = \varsigma$ and thus the complex potential for the fluid flow outside a unit disk will be:

$$\chi(z) = \theta . g(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

4.5 Tilted Plate

Let us next consider the case of a tilted plate in a uniformly horizontal fluid flow. The cross will be the line segment $z(t) = te^{-i\phi}$ where $-1 \leq t \leq 1$ obtained by rotating the horizontal line segment $[-1,1]$ through an angle $-\phi$. The goal is to construct a fluid flow past the tilted segment that is asymptotically horizontal at large distance. The air flow will be going from left to right, ϕ is called the attack angle of the plate or airfoil

CHAPTER 5:

Problems of Conformal Mapping

5.1 Introduction

The present work is concerned with the conformal mapping of one domain of a certain class into another such domain under given auxiliary conditions. Necessary and sufficient conditions for the possibility of such a mapping will be given in terms of modules of the domainsin question. These constitute an extension of the concept of extremal length. The method of extremal length is due to ahlfors and beurling. It may be regarded as a development, on the one hand, of the important method initiated independently by Grotzsch and Ahlfors and on the other of the work of Beurling. It is further related to some considerations of Teichmuller.

This introduction will be devoted to an exposition of known properties of extremal length. The remaining three chapters will contain the solution of mapping problems for the pentagon, the hexagon and the triply-connected domain.

The general concept of extremal length is introduced as follows: let Ω be a domain in the z-plane containing a family Γ of curves γ. Let ρ be a non-negative function defined in Ω of integrable square and such that $\int \gamma\, \rho |dz|$ exists for all γ in Γ with $\int \gamma\, \rho |dz| \geq 1$. Let g.l.b. $\iint \Omega \rho^2 dxdy = 1 / \lambda$ where $= x + iy$, ρ runs through the functions just defined and λ may be ∞. Then λ is called the external length of the family Γ.

This quantity λ is a conformal invariant. Indeed, let Ω be mapped conformally on a domain Ω' in the z' - plane by a function $z' = f(z)$, while $z = \emptyset(z')$. Let the family Γ go into a family Γ' of curves γ'. Then give a function $\rho'(z)$ admissible for Ω and Γ, we obtain a function $\rho'(\mathrm{z}) = \rho(\phi(z'))|\phi'(z')|$.

5.2 The coefficient problem

Let S be the family of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$

Which are regular and schlicht in $|z| < 1$. The most famous problem concerning these functions is whether $|a_n| \leqq n(n = 2,3, \dots)$ with equality for any n only in the case when $f(z) = \frac{z}{(1-\eta z)^2} = z + 2\eta z^2 + 3\eta z^3 + \cdots$, $|\eta| = 1$.

5.2 Examples of Conformal Mapping

Example: H_0 be the half-plane above the line $y = \tan(a)x$

Solution : We do this in two steps. First use the rotation $\mathrm{T}_{-\mathrm{a}}(\mathrm{a}) = \mathrm{e}^{-\mathrm{ia}}\mathrm{z}$ to map H_0 to the upper half-plane. Follow this with the map T_0. So our map is $T_0 \circ T_{-a}(z)$.

Example: A be the channel $0 \leq y \leq \pi$ in the xy-plane. Find a conformal map from A to the upper half-plane.

Solution: The map $f(z) = e^z$ does not trick.

Example: B be the upper half of the unit disk, find a conformal map from B to the upper half plane.

Solution: The map $T_0^{-1}(z)$ maps B to the second quadrant, then multiplying by $-i$ maps this to the first quadrant, then squaring maps this to the upper half plane. In the end we have $f(z) = \left(-i\left(\frac{iz+i}{-z+1}\right)\right)^2$.

Example: A be the infinite well $\{(x, y): x \leq 0, 0 \leq y \leq \pi\}$. Find a conformal map from A to the upper half-plane.

Solution: The map $f(z) = e^z$ maps A to the upper half of the unit disk. Then we can use the map from the half-disk to the upper plane.

<u>CHAPTER 6</u> :

Conclusion

Initially, this paper reviews the literature on conformal mapping methods in the research of mechanical analysis in tunnel engineering. Three methods have been developed or adopted in these studies: optimization, iteration and extended Melentiev's methodology. These methods are erroneously thought to be effective in conformally mapping noncircular tunnel sections conformally. Through a detailed analysis and scrutiny, these methods turn out to be only the ordinary transformation function between the boundaries. More importantly, it is proven that these methods can not guarantee the conformality of the mapping between rock cavity sections and circular domains according to the definition of conformal mapping, the existence and uniqueness theorem and the boundary correspondence principle.

Conformal mapping is a powerful tool in complex analysis and engineering, used to transform complex shapes while preserving angles between curves. It simplifies complex problems by mapping them to more convenient domains, and plays a crucial role in fields like fluid dynamics, electrostatics, and heat transfer. Conformal mapping is most effective when dealing with two-dimensional problems and when the geometry has a high degree of symmetry. It can be more challenging to apply in situations where symmetry is broken.

Consequently, the evoluation of conformal mapping reseach in the complex analysis field is investigated thoroughly. The advanced SC formulas for multiply connected domains and related numerical methods are identified and chosen to develop the new procedure. By extending the discretization function for boundaries, the conformal mapping procedure for tunnels with polygonal boundaries, multi-arc boundaries, smooth curve boundaries and mixed boundaries is proposed and programmed. Then, conformal mapping and inverse mapping of six-typical tunnel cross sections buried at depth including rectangular holes, multiarc tunnels, horseshoe-shaped tunnels and multiple holes are performed and illustrated in case 1 to case 6. The conformal mapping of the shallow cavities can be implemented using the symmetry principle of conformal mapping, as illustrated in case 7.

Finally tips on applying this procedure to the mechanical analysis study of irregularly shaped cavities are presented. The approximate conformal mappings of the rock domains in case 3 and case 4 are performed. The approximation error is significantly reduced by increasing the item number when approximating the conformal mapping based on a finite number of items of the power series. The contribution of this paper will be used not only in deriving analytical solutions of tunnels but also in the back analysis of tunnel mechanical behavior, which will be of great use for increasing the safety of tunnel engineering.

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