A STUDY ON FUNDAMENTAL SOLUTION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Dissertation submitted to the Department of Mathematics in fulfilment of the requirements for the award of the degree of Master of Science

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CONTENT

<u>CERTIFICATE</u>

This is to certify that **GOLAM OSMANI** bearing **Roll No MAT-23/23** and **Regd. No. MSSV-0023-101-001351** has prepared his dissertation entitled "**FUNDAMENTAL SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS**" submitted to the Department of Mathematics, **MAHAPURUSHA SRIMANTA SANKARADEVA VISWAVIDYALAYA**, Nagaon, for fulfilment of MSc. degree, under guidance of me and neither the dissertation nor any part thereof has submitted to this or any other university for a research degree or diploma.

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DECLARATION

I, **GOLAM OSMANI**, bearing ROLL No- **MAT-23/23** student of final semester, Department of Mathematics , **MAHAPURUSHA SRIMANTA SANKARADEVA VISHWAVIDYALAYA** (MSSV), do hereby declare that the work incorporated in this dissertation entitled "**FUNDAMENTAL SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS**", for the award of degree of Master of Science in Mathematics, has been carried out and interpreted by me under the supervision of **DR. MIRA DAS** , Assistant Professor, Department of Mathematics , (MSSV) Nagaon. This dissertation is original and has not been submitted by me for the award of degree of diploma to any other University or Institute. I have faithfully and accurately cited all my sources, including books, journals, handouts and unpublished manuscripts, as well as any other media, such as the internet, letters or significant personal communication.

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Yours' sincerely
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ABSTRACT

This dissertation investigates the construction and analysis of Fundamental Solution of Linear Partial Differential Equation (PDEs), which play a central role in the theory and application of mathematical physics, engineering, and applied mathematics.

A fundamental solution provides a powerful tool for representing solutions to linear PDEs via convolution with source terms, offering deep insight into the behavior of physical systems governed by differential laws.

The work begins with a detailed theoretical framework for linear constant-coefficient PDEs, emphasizing classical examples such as the Laplace, heat, and wave equations. The existence and uniqueness of fundamental solutions are examined through Fourier transform methods and distribution theory. For variable-coefficient linear operators, we extend the analysis using parametrix constructions and microlocal techniques to study singularity propagation and asymptotic properties.

Additionally, the dissertation addresses the regularity and support properties of fundamental solutions in both Euclidean space and bounded domains, highlighting their connections with Green's functions and boundary value problems. Applications to inverse problems, control theory, and numerical approximation are also discussed, demonstrating the practical relevance of the theoretical findings.

This work contributes to the broader understanding of PDE theory by providing refined estimates and structural insights into the nature of fundamental

solutions, thereby supporting further developments in mathematical modeling and analysis.

Let me know the specific PDEs or mathematical methods used in your work, and I can tailor it even more precisely.

LITERATURE REVIEW

To understand the development of fundamental solutions in partial differential equations (PDEs) and the theory of generalized functions, I studied several important books and research papers written by well-known mathematicians. These works helped me trace how the field has evolved from classical theory to modern approaches.

The earliest work I studied was by E.E. Levi (1907). He made some of the first attempts to understand fundamental solutions of linear PDEs, especially for elliptic equations. His contributions laid the base for many future discoveries in the area.

Books by M.D. Raisinghania, such as Advanced Differential Equations and Ordinary and Partial Differential Equations, were very helpful. These books explain the theory in a very clear and step-by-step manner, which is good for students like me. They cover both basic and advanced topics in PDEs, making it easier to build strong foundational knowledge.

A.N. Krylov's (1933) book Differential Equations of Mathematical Physics is an important historical work. It connects PDEs with physical problems in mechanics and physics. This book helped me see the link between pure mathematics and real-world applications.

Petrovsky (1938) worked on proving the existence of fundamental solutions for higher-order PDEs. His research extended Levi's earlier work and gave mathematicians more tools to deal with complex equations.

A major turning point came with L. Schwartz's Theory of Distributions (1950). He introduced the idea of "distributions" or generalized functions. This allowed mathematicians to solve differential equations in a much broader sense, especially when classical solutions did not exist.

L. Ehrenpreis (1954) made significant contributions by solving division problems related to distributions. His results helped in extending the Schwartz theory.

Together with Malgrange, they proved what is now called the Malgrange-Ehrenpreis Theorem (1955). This important theorem says that every linear PDE with constant coefficients has a fundamental solution in the sense of distributions.

In physics, N.N. Bogolubov and D.V. Shirkov (1976) used the theory of generalized functions in their book Introduction to Quantum Field Theory. Their work showed how mathematics and physics are deeply connected, especially in dealing with singularities and wave behavior.

J.F. Colombeau (1985) proposed a new kind of generalized function that could handle nonlinear operations better than Schwartz's theory. His approach gave a solution to many problems where the traditional distribution theory was limited.

Later researchers like Yu V. Egorov (1990) and A.B. Antonevich & Ya V. Radyno (1991) continued developing new ways to understand and construct generalized functions. They provided methods that make it easier to work with differential equations involving singularities or non-smooth data.

Finally, the work of Ya V. Radyno and Fu Than Ngo (1993) explored how differential equations can be treated in the algebra of new generalized functions. Their approach brought a fresh perspective on solving PDEs where standard methods fail.

INTRODUCTION

A fundamental solution for a linear partial differential equation (PDE) is a particular solution that, when convolved with an arbitrary function, yields a solution to the PDE with that arbitrary function as the source. In essence, it's a way to build solutions to the inhomogeneous PDE by finding a solution as the source.

A fundamental solution, also known as a Green's function, is a distribution (a generalized function) that satisfies the PDE with a Dirac delta function as the forcing term on the right-hand side.

This means the solution has a singularity at specific point usually the origin, but it's a "point" solution that can be used build up solutions for other functions on the right-hand side.

Why it's important:

Solving inhomogeneous equation:

Fundamental solutions are crucial for solving PDE's where the right – hand side (forcing term) is not zero (inhomogeneous equations).

Building up solutions:

By using convolution with the fundamental solution, one can find the solution for any function on the right-hand side of the PDE.

Method of fundamental solutions (MFS):

This numerical method uses fundamental solutions to approximate solutions of PDE’s, particularly when dealing with complex geometries or non-standard boundary conditions according to an article on word Scientific Publishing.

Relationship to Green’s functions:

Fundamental solutions are closely related to Greens functions, which are used in solving boundary value problems.

In simpler terms:

Imagine you have a specific equation, and you want to find solutions for different inputs. A fundamental solution is a special solution that lets you “buildup” the solutions for those different inputs by using a combination of the fundamental solution and other solutions.

Key concepts:

Dirac delta function:

A mathematical function that is zero everywhere except at a single point, where it has an infinite value.

Convolution:

A mathematical operation that combines two functions to produce a third function, often used to solve PDE’S.

Distribution:

A generalized function that can represent a function with singularities or discontinuities.

Linear PDE s have highly interdisciplinary aspects. For finding applications of divers field like Physics, engineering, biology, and even finance.

In this field, so many researches expressed their own view and also completed their research in different aspect. We have studied some of them. A.N. Krylov studied in his research and he visualized as Differential Equations of Mathematical Physics,1933

CHAPTER-1

Introduce to PDEs and fundamental Solutions.

1.1 Introduce the concept of a fundamental solutions.

A partial differential equation (PDE) is a mathematical equation involving multiple independent variables, an unknown function of those variables, and its partial derivatives. A fundamental solution to a PDE is a specific solution, often singular, that acts as a building block for constructing more general solutions. It represents the response of a system to a concentrated impulse or point source.

1.1.1 Partial Differential Equations (PDEs)

- PDEs are used to model a wide range of phenomena in various fields like physics, engineering, and finance.
- They involve an unknown function (dependent variable) and its partial derivatives with respect to multiple independent variables.
- The order of a PDE is determined by the highest order of the partial derivatives present in the equation.
- Examples of PDEs include the heat equation, wave equation, and Laplace's equation.

1.1.2 Fundamental Solution

- A fundamental solution is a special solution to a PDE, often involving a Dirac delta function as a source term.
- It represents the response of the system to a localized impulse or a point source.

- Think of it as the "simplest" solution, from which more complex solutions can be built.
- The fundamental solution to a PDE can be used to solve the equation with more general source terms and boundary conditions.
- For example, the fundamental solution of the heat equation can be used to find the temperature distribution in a region with a given heat source.

1.2 Explain why fundamental solutions are important for solving PDEs.

Fundamental solutions are crucial for solving partial differential equations (PDEs) because they provide a building block for constructing more complex solutions. By convolving the fundamental solution with a source term, one can solve a variety of PDEs, particularly those with complicated source terms or boundary conditions. This approach is widely used in physics and engineering to solve problems described by PDEs.

1.2.1 More detailed explanation of Fundamental solutions

What are fundamental solutions?

A fundamental solution of a PDE is a solution corresponding to a concentrated "point source" or "impulse". For example, in the context of Laplace equation ($\nabla^2 u = 0$), the fundamental solution represents the Potential created by a point charge.

1.2.2 Green's Function Approach

A fundamental l(one) solution, often denoted as $\emptyset$, is a solution to a PDE where the source term is a Dirac delta function. This means it represents the response of a system to a point source.

1.2.3 Solving Complex problems

Many physical phenomena are described by PDEs with complex geometries or boundary conditions. Fundamental solutions allow you to address these complexities by breaking them down into simpler, solvable components.

1.2.4 Numerical Methods

The method of fundamental solutions (MFS) is a boundary-type Numerical method that relies on fundamental solutions to approximate Solutions to PDEs. This method is particularly useful for problems Where analytical solutions are difficult to obtain.

Examples in physics

Fundamental solutions are used in various fields:

- **Electromagnetism:** Solving Maxwell's equations.
- **Fluid Dynamics:** Solving the Navier-Stokes equations.
- **Quantum Mechanics:** Solving the Schrodinger equation.

1.3 Introduce the Dirac delta function and its role in defining fundamental solutions.

The Dirac delta function, often denoted as $\delta(x)$, is a mathematical construct that plays a crucial role in various fields such as physics, engineering, and applied mathematics. It's not a function in the traditional sense, but rather a generalized function or distribution, which is used to model point sources or impulsive forces. Let's break it down and then explore its role in defining fundamental solutions to differential equations.

1.3.1 What is the Dirac delta function?

The Dirac delta function is defined by two key properties:

(i) **Localization:** It's zero everywhere except at $x = 0$, where it is infinitely concentrated mathematically:

$$\delta(x) = 0 \text{ for } x \neq 0.$$

(ii) **Integral Property:** The delta function is defined such that the integral over the entire real line equals 1. In other words:

$$\int_{-\infty}^{\infty} \delta(x) dx = 1.$$

This is an essential property because it implies that the delta function "captures" a unit of area in an infinitesimally small region at $x = 0$.

It is often described informally as an infinitely narrow and tall spike at $x = 0$, with area 1.

1.3.2 Mathematical Representation

One of the ways to think about the delta function is through it's limiting behaviour. For example, a sequence of functions like the Gaussian function:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi\epsilon}} e^{\frac{-x^2}{\epsilon}}$$

can be seen as an approximation of the delta function in the limit as

$\epsilon \to 0$, where the function becomes increasingly narrow and tall,

concentrating all of its mass at $x = 0$.

1.3.3 Role in Defining Fundamental Solution

In the context of differential equations, the Dirac delta function plays a crucial role in representing impulses, point sources, or discontinuities. It's widely used in defining the fundamental solutions to certain classes of differential equations. Let's break this down further:

1.3.4 Green's Function

A fundamental solution to a linear differential operator (such as α) is often referred to as a Green's function. Green's function is a solution to the equation

$$\alpha G(x,x') = \delta(x - x')$$

where α is a differential operator, and $G(x,x')$ is the Green's function $G(x,x')$ describes the response of the system to that point source. In other words, The Dirac delta function helps us understand how a system responds to a localized impulse. For example, in the context of heat conduction, it represents how a system responds to a sudden point heat source.

1.3.5 Solution to Differential Equations

In physics, the Dirac delta function is used in the formulation of initial conditions for differential equations. For example, in the case of the wave equation, the delta function can describe an impulse applied at a single point in space at a single instant in time.

1.3.6 Convolution with other Functions

The delta function has a special property when it comes to convolution. When convolved with another function $f(x)$, it acts as an identity:

$$f(x) * \delta(x) = f(x).$$

This property makes the delta function incredibly useful in solving differential equations, as it allows for the transformation of differential operators into algebraic ones in the context of integral equations.

CHAPTER 2

First – Order PDEs

2.1 Discuss the method of characteristics for solving first-order PDEs.

The method of characteristics is technique for solving first-order partial differential equations (PDEs) by transforming them into a system of ordinary differential equations (ODEs) along the characteristic curves. These curves are special paths in the space of independent variables where the PDE effectively becomes an ODE. Solving these ODEs provides information about the solution of the original PDE.

2.1.1. Identifying the PDE and characteristic Curves

- Consider a first-order quasilinear PDE:

$$\boldsymbol{a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u).}$$

- The coefficients a, b and c are functions of the independent variables x and y, and the dependent variable u.
- The method introduces characteristic curves, defined by the system of ODEs:

    I. $\frac{dx}{dt} = a(x,y,u)$

    II. $\frac{dy}{dt} = b(x,y,u)$

    III. $\frac{du}{dt} = c(x,y,u)$

- The solution of these ODEs gives $x(t), y(t)$ and $u(t)$, which represent the characteristic curves in the $x - y$ plane and the corresponding values of u along those curves.

2.1.2 Transforming the PDE into ODEs

- The original PDE is expressed in terms of $\frac{du}{dt}, \frac{dx}{dt}$ and $\frac{dy}{dt}$, where t is a parameter along the characteristic curve.
- By comparing coefficients in the PDE along the expressions for $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{du}{dt}$, we can establish a system of ODEs.

2.1.3 Solving the ODE System

- The system of ODEs is solved along the characteristic curves, typically using initial conditions given for the PDE.
- The solutions of the ODEs provide the values of u along these curves, which can then be used to find the solution of the original PDE.

2.1.4 Finding the solution to the PDE

The solutions obtained along the characteristic curves are used to construct the general solution of the PDE.

The method essentially allows us to “track” the solution along these special curves, simplifying the problem from a PDE to a series of ODEs.

In essence, the method of characteristics provides a way to convert a first-order PDE into a set of ODEs that can be solved along specific curves (the characteristic), ultimately leading to the solution of the original PDE.

2.2 Introduce the concept of characteristics curves and their role in finding solutions.

Characteristic curves are a powerful tool in solving certain

Types of partial differential equations (PDEs) by transforming them into ordinary differential equations (ODEs) along these curves. These curves, derived from the PDE itself, define paths in the solution space along which the PDE behaves like an ODE, simplifying the solution process. By solving the ODEs along these curves, one can then relate the solution back to the original PDE.

2.2.1 What are characteristic curves?

In the context of PDEs, characteristic curves are special curves in the

solution space along which the PDE behaves like a simpler ODE. They are essentially solutions to a system of ODEs derived from the original PDE.

2.2.2 Role in Finding Solutions

(i) Finding the Curves:
The method of characteristics involves finding these special curves by solving a system of ODEs derived from the original PDE. These ODEs are called the characteristic equations.

(ii) Solving along the curves:
Once the characteristic curves are found, the PDE reduces to a set of ODEs. These ODEs can then be solved along each characteristic curve. This solution is often constant or follows a simple relationship along the curve.

(iii) Reconstructing the Solution:
By combining the solutions found on different characteristic curves, we can reconstruct the solution to the original PDE.

2.3 Explain how the method of characteristics can be used to solve problems with initial conditions.

The method of characteristics is a technique used to solve first-order

partial differential equations (PDEs) by transforming them into a system of ordinary differential equations (ODEs) along specific curves called characteristics. These curves are determined by the

2.3.1 Identify the PDE and Initial Condition

Coefficients of the PDEs and when combined with initial conditions, allow

for the solution to be found. Start with a first-order PDE, typically in two

independent variables (x, t) and a given initial condition.

For Example PDE:

$$a(x,t)\frac{\partial u}{\partial x} + b(x,t)\frac{\partial u}{\partial t} = c(x,t)$$

Initial Condition: $u(x, 0) = f(x)$

2.3.2 Define Characteristic Equations

- The method introduces a new parameter ‘s’ (often called characteristic variable) and defines characteristic curves in the x-t plane using a system of ODEs:

$$\frac{dx}{dt} = a(x,t)$$

$$\frac{dt}{ds} = b(x,t)$$

- The PDE is then transformed into an ODE along these curves using the chain rule:

$$\frac{du}{ds} = c(x,t)$$

2.3.3 Solve the ODEs

- Solve the characteristic equations ($\frac{dx}{ds}$ and $\frac{dt}{ds}$) to find expressions for $x(s)$ and $t(s)$ in terms of the parameter 's'. These expressions define the characteristic curves.
- Solve the equation for $\frac{du}{ds}$ to find $u(s)$ along these curves. Since u is constant along the characteristic if $c(x,t)$ is zero, this step can be simplified.

2.3.4 Apply Initial Conditions:

Use the initial condition ($u(x,0) = f(x)$) to relate the initial point on the characteristic curve to the solution. For Example, If the initial condition is given at $t = 0$, then the initial point on the characteristic curve will have $t = 0$.

The value of u at his point will be $f(x\_0)$, where x_0 is the initial x value. The solution $u(s)$ along the characteristic curve can then be expressed in terms of the initial value $f(x,0)$

2.3.5 Find the Solution in Original Co-ordinates:

(i) Express x and t in terms of s using the solutions for $\frac{dx}{ds}$ and $\frac{dt}{ds}$

(ii) Substitute these expressions for x and t back into the solution for $u(s)$ to obtain $u(x,t)$ in terms of the original variables.

<u>Solution</u> (i)

To express x and t in terms of a parameter (usually s), using the given differential equations: $\frac{dx}{dt}$ and $\frac{dt}{ds}$

We can use the chain rule:

$$\frac{dx}{ds} = \frac{dx}{dt} \cdot \frac{dt}{ds}$$

Step I: Integrate to find $t(s)$

Given $\frac{dt}{ds}$, integrate to find

$$t(s) = \int \frac{dt}{ds} ds + c_t$$

Where c_t is an integration constant.

Step II: Integrate to find $x(s)$

Given $\frac{dx}{ds}$, integrate to find

$$x(s) = \int \frac{dx}{ds} ds + c_x$$

Where c_x is an integration constant.

CHAPTER 3

Second-Order PDEs

3.1 Introduction:

An equation is said to be of order two, if it involves at least one of the differential coefficients-

$r = \left(\frac{\partial^2 z}{\partial x^2}\right), s = \left(\frac{\partial^2 z}{\partial x \partial y}\right), t = (\frac{\partial^2 z}{\partial y^2})$, but now of higher order; the quantities p and q may also enter into the equation. Thus the general form of a second order partial differential equation is:

$$f(x, y, z, p, q, r, s, t) = 0 \rightarrow (1)$$

The most general linear partial differential equation of order two in two independent variables x and y with variable coefficient is of the form:

$$Rr + Ss + Tt + Pp + Qq + Zz = F \rightarrow (2)$$

Where R, S, T, P, Q, Z, F are functions of x and y only and not all R, S, T are zero.

Example (i)

Solve $r = 6x$.

Solution: The given equation can be written as $\frac{\partial^2 z}{\partial x^2} = 6x$ ……. (1)

Integrating of (1) both side w.r.to x we get:

$$\frac{\partial z}{\partial x} = 3x^2 + \varphi_1(y) \qquad \text{……… (2)}$$

Where $\varphi_1(y)$ is an arbitrary function of y.

Again integrating of (2) both side w.r.to x we get:

$$z = x^3 + x\varphi_1(y) + \varphi_2(y)$$

Where $\varphi_2(y)$ is an arbitrary function of y

Example (ii)

Solve $ar = xy$

Solution: Given equation can be written as $\frac{\partial^2 z}{\partial x^2} = \frac{1}{a}xy$ ……… (1)

Integrating (1) w.r.to x we get

$$\frac{\partial z}{\partial x} = (\frac{y}{a})\frac{x^2}{2} + \varphi_1(y) \quad \text{……… (2)}$$

Where $\varphi_1(y)$ is an arbitrary function of y

Integrating (2) w.r.to x,

$$z = \left(\frac{y}{a}\right)\frac{x^3}{6} + x\varphi_1(y) + \varphi_2(y)$$

Where $\varphi_2(y)$ is an arbitrary function of y.

3.2 Partial differential equations with constant coefficient:

We know that the general from of a linear partial differential equation

$$A_n \frac{\partial^n z}{\partial x^n} + A_{n-1} \frac{\partial^n z}{\partial x^{n-1}\partial y} + A_{n-2} \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + A_1 \frac{\partial^n z}{\partial y^n} = f(x, y) \text{... (1)}$$

Where the coefficients $A_n, A_{n-1}, A_{n-2}, \dots, A_1$ are constants or functions of x and y.

If $A_n, A_{n-1}, A_{n-2}, \dots, A_1$ are all constants, then (1) is called a linear partial differential equation with constant coefficients. We denote $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ by D (or D_x) and D' (or D_y) respectively.

Therefore (1) can be written as:

$$[A_n D^n + A_{n-1} D^{n-1} D' + A_{n-2} D^{n-2} D'^2 + \cdots + A_1 D'^n]z = f(x, y)$$

…….. (2)

Or $\varphi(D, D')z = f(x, y)$

The complementary function of (2) is given by

$$[A_n D^n + A_{n-1} D^{n-1} D' + A_{n-2} D^{n-2} D'^2 + \cdots + A_1 D'^n]z = 0$$

……… (3)

Or $\varphi(D, D')z = 0$

Let $z = F(y + mx)$ be the part of the solution

$$Dz = \frac{\partial z}{\partial x} = mF'(y + mx)$$

$$D^2 z = \frac{\partial^2 z}{\partial x^2} m^2 F''(y + mx)$$

… … …

… … …

$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n F^n(y + mx)$$

And

$$D'z = \frac{\partial z}{\partial y} = F'(y + mx)$$

$$D'^2 z = \frac{\partial^2 z}{\partial y^2} = F''(y + mx)$$

… … …

… … …

$$D'^n z = \frac{\partial^n z}{\partial y^n} = F^n(y + mx)$$

Substitute these values (3), we get

$$[A_n m^n + A_{n-1} m^{n-1} + A_{n-2} m^{n-2} + \cdots + A_1] F^n(y + mx) = 0$$

Which is true if 'm' is a root of the equation

If $m_1, m_2, \dots, m_n$ are distinct roots, then complementary function is

$$z = \varphi_1(y + m_1 x) + \varphi_2(y + m_2 x) + \cdots + \varphi_n(y + mx)$$

Where $\varphi_1, \varphi_2, \dots, \varphi_n$ are arbitrary functions.

$$\therefore \varphi(D, D')z = 0$$

We replace D by m and D' by 1 to get the auxiliary equation from which we get roots.

3.3 Linear partial differential equation with constant coefficients:

Homogeneous and non homogeneous linear equations with constant coefficients

A partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients being constants or functions of x and y, is known as a linear partial differential equation.

The general form of such an equation is-

$$\left[A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + A_n \frac{\partial^n z}{\partial y^n}\right] + [B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}}$$

$$+B_1 \frac{\partial^{n-1} z}{\partial x^{n-2} \partial y} + B_2 \frac{\partial^{n-1} z}{\partial x^{n-3} \partial y^2} + \cdots + B_n \frac{\partial^{n-1} z}{\partial y^{n-1}}] + [M_0 \frac{\partial z}{\partial x} + M_1 \frac{\partial z}{\partial y}] + N_0 z = f(x, y) \text{.....} (1)$$

Where the coefficients $A_0, A_1, \dots, A_n, B_0, B_1, \dots, B_n, M_0, M_1$ and N_0 are all constants, then (1) is called a linear partial differential equation with constant coefficients.

For convenience $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ will denoted by D and D' respectively.

Example (i) Solve $(D^2 + 3DD' + 2D'^2)z = x + y$

Solution:

The auxiliary equation of the given equation is $m^2 + 3m + 2 = 0$.

So that $m = -1, -2$

Therefore, C. F= $\emptyset_1(y - x) + \emptyset_2(y - 2x)$,

Where $\emptyset_1, \emptyset_2$ being arbitrary functions

Now, $\text{PI} = \frac{1}{D^2+3DD'+2D'^2}(x + y)$

$= \frac{1}{1^2+3\cdot1\cdot1+2\cdot1^2} \iint v dv dv$, where $v = x + y$

$= \frac{1}{6}\int \frac{v^2}{2} dv$

$= \frac{1}{6}\frac{v^3}{6}$

$= \frac{1}{36}(x + y)^3$

Hence the required general solution is

$z = C.F + PI$

$$\therefore z = \emptyset_1(y - x) + \emptyset_2(y - 2x) + \frac{1}{36}(x + y)^3$$

Example (ii) Solve $(D^2 - a^2D'^2)z = x$

Solution: Here auxiliary equation is $m^2 - a^2 = 0$

So that, $m = a, -a$

Therefore C.F= $\emptyset_1(y + ax) + \emptyset_2(y - ax)$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions.

Now, PI= $\frac{1}{D^2 - a^2 D'^2}(x)$

$= \frac{1}{D^2[1 - a^2(\frac{D'^2}{D^2})}(x)$

$= \frac{1}{D^2}\left[1 - a^2\left(\frac{D'^2}{D^2}\right)\right]^{-1}$ (x)

$= \frac{1}{D^2}[1 + a^2(D'^2/D^2) + \cdots]x$

$= \frac{1}{D^2}x$

$= \frac{x^3}{6}$

Hence the required solution is $z =$ C. F+PI

$$z = \emptyset_1(y + ax) + \emptyset_2(y - ax) + \frac{x^3}{6}$$

3.4 Classification of second order partial differential equation

Definition: A second order partial differential equation which is linear w.r. to the second order partial derivatives i.e. r, s and t is said to be a quasi linear PDE of second order.

For example the equation:

$Rr + Ss + Tt + f(x, y, z, p, q) = 0$ ………………….. (1)

Where $f(x, y, z, p, q)$ need not be linear, is a quasilinear partial differential equation. Here the coefficients R, S, T may be functions of x and y, however for the sake of simplicity we assume them to be constants.

The equation (1) is said to be

i. Elliptic if $S^2 - 4RT < 0$

ii. Parabolic if $S^2 - 4RT = 0$

iii. Hyperbolic if $S^2 - 4RT > 0$

CHAPTER 4

Laplace Equation

4.1 Introduction:

The Laplace equation is a second-order partial differential equation that arises frequently in various fields of physics and mathematics. It describes the behavior of harmonic functions and is often written as $\nabla^2 u = 0$. Where ∇^2 is the Laplace operator and u is a scalar function. It's fundamental equation in potential theory and has applications in areas like electrostatics, gravitation, fluid dynamics, and heat conduction.

4.2 Introduce the Laplacian operator and its properties:

The Laplacian operator, often denoted as ∇^2 or Δ is a second order differential operator that measures the local curvature of a function. It's essentially the divergence of the gradient of a scalar field. In simpler terms, it tells you how much a functions value at a point differs from the average value of the function in the immediate neighbourhood of that point.

4.2.1 Properties of the Laplacian Operator:

(i) Linearity:

The Laplacian operator is linear, meaning that $\nabla^2(af + bg) = a\nabla^2 f + b\nabla^2 g$, where 'a' and 'b' are constants and 'f' and 'g' are scalar functions.

(ii) Second-Order Differential Operator:

It involves second-order partial derivatives, making it a second-order differential operator.

(iii) Harmonic Function:

Solutions to the Laplace equation are known as harmonic functions.

(iv) Boundary Value Problem:

The Laplace equation is often encountered in boundary value problems, where the solution is sought within a region with specified values on the boundaries.

4.3 Discuss the principle of superposition for solving Laplace's equation:

The principle of superposition is a powerful tool used to solve Laplace's equation, especially in the context of linear boundary value problems.

Laplace's equation in its standard form is:

$$\nabla^2 \emptyset = 0$$

Where $\emptyset$ is a scalar potential function and ∇^2 is the Laplacian operator. The principle of superposition applies because Laplace's equation is linear and homogeneous.

4.3.1 What is the principle of superposition?

The principle of superposition states that:

If $\emptyset_1$ and $\emptyset_2$ are solutions to Laplace's equation, then any linear combination $\emptyset = a\emptyset_1 + b\emptyset_2$ is also a solution, where a and b are constants.

Mathematically:

$$\nabla^2 \emptyset_1 = 0, \quad \nabla^2 \emptyset_2 = 0 \quad \Longrightarrow \quad \nabla^2 (a\emptyset_1 + b\emptyset_2) = 0$$

This follows because the Laplace operator is linear:

$$\nabla^2(a\emptyset_1 + b\emptyset_2) = a\nabla^2\emptyset_1 + b\nabla^2\emptyset_2 = 0$$

4.3.2 Why It's useful in solving Laplace's equation

In practical problems especially in electrostatics, fluid flow, or heat conduction the boundary conditions can be complex. The superposition principle allows us to:

4.4 Explain the concept of Harmonic functions and its Properties:

Harmonic functions are a class of twice-continuously differentiable functions that satisfy Laplace's equation. In simpler terms, a function is harmonic if its Laplacian (a mathematical operator) is equal to zero. They are fundamental in various fields of mathematics, physics, and engineering.

Definition:

A function $u(x, y)$ is harmonic if it satisfies the following partial differential equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This equation is also known as Laplace's equation.

Properties:

Mean Value Property:

A harmonic function's value at any point is equal to the average of its values over any circle (in 2D) or sphere (in 3D) centred at that point. This means that the function's value at the centre of a circle or sphere is the average of its values on the circumference or surface.

Maximum Principle:

A non-constant harmonic function cannot attain its maximum or minimum value at an interior point of its domain. The maximum and minimum values are always attained on the boundary of the domain.

Harmonic Conjugates:

For a harmonic function $u(x, y)$in a region, there exists another harmonic

function $v(x, y)$ (called its harmonic conjugate) such that the complex function

$f(z) = u(x, y) + iv(x, y)$ is analytic (has a derivative at every point).

CHAPTER 5: Heat Equation

5.1 Introduction

The heat equation is a partial differential equation that describes how temperature (or a similar quantity) distributes over time and space within a given material. It is fundamental to understanding heat transfer and is used in various scientific and engineering fields. The equation is parabolic and often expressed as $\frac{\partial u}{\partial t} = \alpha \nabla^2 u$, where '$u$' represents temperature, 't' is time, 'α' is the thermal diffusivity, and ∇^2 is the Laplacian operator.

Here's a more detailed explanation:

What it describes: The heat equation models how temperature changes over time in response to differences in temperature within a region. It essentially says that the rate of temperature change at a point is proportional to the curvature of the temperature distribution around that point.

5.2 Introduce the Heat equation and its physical interpretation

The heat equation is a partial differential equation that describes how temperature (or concentration of a substance) distributes over time in a given region. It essentially models the diffusion of heat or other quantities, showing how they spread from areas of high concentration to areas of low concentration over time. Physically, it can be used to understand how heat flows through materials, how pollutants disperse, or even how information spreads.

Mathematical Formulation:

The heat equation in its simplest form, for one spatial dimension (like a thin rod) is:

$$\frac{\partial u}{\partial t} = \alpha\, \partial^2 u/\partial x^2$$

Where, $u(x,t)$: represents the temperature (or concentration) at position x and time t. α: (alpha) is the thermal diffusivity (or diffusion coefficient), a constant that depends on the material properties and indicates how quickly heat diffuses.

$\partial u/\partial t$: represents the rate of change of temperature with respect to time.

$\partial^2 u/\partial x^2$: represents the second spatial derivative of temperature, which essentially measures the curvature of the temperature profile.

5.2.1 Physical Interpretation:

Diffusion:

The heat equation models the process of diffusion, where heat (or any quantity) tends to spread out from regions of high concentration to regions of low concentration.

Temperature Distribution:

It describes how the temperature changes over time in a given region, given some initial temperature distribution and boundary conditions (e.g., fixed temperatures at the ends of a rod).

Rate of Change:

The equation states that the rate of temperature change at a point is proportional to the curvature of the temperature profile at that point.

A positive curvature (concave up) means the temperature is increasing, and a negative curvature (concave down) means the temperature is decreasing.

Thermal Diffusivity:

The parameter α (alpha) controls how quickly heat diffuses. A larger α means heat spreads out faster.

Applications:

The heat equation has broad applications beyond just heat transfer, including:

Pollution Dispersion: Modeling how pollutants spread in air or water.

Financial Modeling: Some variants of the heat equation are used in financial mathematics.

Geometric Modeling: The heat equation can be used to smooth or fair geometric shapes.

Image Processing: It can be used to blur or sharpen images.

In essence, the heat equation provides a mathematical framework for understanding how quantities spread and equilibrate over time, making it a powerful tool in many scientific and engineering disciplines.

5.3 Explain the concept of the heat kernel and its role in finding solutions.

The heat kernel is a fundamental concept in mathematical physics, differential geometry, and partial differential equations (PDEs), particularly in solving the heat equation.

5.3.1 What is the heat kernel:

The heat kernel is the fundamental solution to the heat equation:

$$\frac{\partial u}{\partial t} = \Delta u$$

Where, $u(x,t)$ is the temperature at point x and time t,

Δ is the Laplace operator, which varies based on the domain (Euclidean space, manifold, etc.)

The heat kernel, usually denoted $k(x,y,t)$, is a fundamental solution to this equation. It represents the temperature at point x at time t, resulting from an initial heat source (i.e., a delta function) located at point y at time 0.

5.3.2 Role of the Heat Kernel in Solving the Heat Equation:

If you know the heat kernel $K(x,y,t)$, the solution $u(x,t)$ to the heat equation with initial condition $u(x,0) = f(x)$ can be written as:

$$u(x,t) = \int K(x,y,t)f(y)dy.$$

This is a convolution of the initial temperature distribution with the heat kernel. It expresses how the initial heat profile $f(y)$ spreads over space and time via the kernel.

5.3.3 Example in Euclidean Space:

In $\mathbb{R}^n$, the heat kernel is explicitly:

$$K(x, y, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}.$$

This is a Gaussian centred at y that broadens over time, illustrating diffusion.

5.4 Discuss the properties of heat equation solutions.

The heat equation solution exhibits several key properties: smoothness, smoothing effect, maximum/minimum values, linearity, and the principle of superposition. Solutions are always smooth (infinitely differentiable) functions, even if the initial condition is not. The equation also causes heat distribution to even out over time, meaning local temperature variations tend to dissipate.

Furthermore, the heat equation solution has a smoothing effect, where sharp edges or peaks in the initial temperature distribution are gradually eroded.

The solutions also satisfy the maximum principle, meaning that the maximum and minimum values of the solution at any given time are bounded by the maximum and minimum values of the initial condition. Lastly, the heat equation is linear, meaning that if you have two solutions, any linear combination of those solutions is also a solution.

5.4.1 Here's a more detailed explanation of each property

Smoothness:

The heat equation solution is infinitely differentiable with respect to both space and time. This means that even if the initial temperature distribution is not smooth (e.g., a step function), the solution will be a smooth, continuous function.

Smoothing effect:

The heat equation describes how heat diffuses over time. As a result, sharp peaks or dips in the initial temperature distribution tend to get smoothed out as heat flows from hotter areas to colder areas.

Maximum/Minimum:

The maximum and minimum values of the solution at any given time are bounded by the maximum and minimum values of the initial temperature distribution. This is known as the maximum principle.

Linearity:

The heat equation is a linear partial differential equation. This means that if you have two solutions, their sum (or any linear combination) will also be a solution.

Superposition:

Because the heat equation is linear, the principle of superposition applies. This means that if you have multiple heat sources, you can find the overall solution by summing the solutions due to each individual source. If u_1 and u_2 are solutions and c_1, c_2 are constants, then $u = c_1 u_1 + c_2 u_2$ is also a solution.

These properties are crucial for understanding how heat distributes in various physical systems and for developing methods to solve the heat equation for different scenarios.

CHAPTER 6 CONCLUSION

A fundamental solution of a linear partial differential equation is a generalized solution (often involving distributions like the Dirac delta function) that represents the response of the PDE to a point source.

It serves as a building block for constructing solutions to more complex problems via convolution with the source them.

For a linear PDE operator L, a fundamental solution $E(x)$ satisfies:

$L(E) = \delta(x)$,

Where $\delta(X)$ is the Dirac delta function.

Once the fundamental solution is known, the solution $u(x)$ to the inhomogeneous PDE $L(u) = f(x)$ can often be written as:

$$u(x) = (E * f)(x)$$

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