“AN INTRODUCTION TO THE p-ADIC NUMBER”

Dissertation submitted to the Department of Mathematics in partial fulfillment of the requirements for the award of the degree of Master of Science in Mathematics

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DISSERTATION

on

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Certificate

This is to certify that the dissertation entitled **"An Introduction To The p-Adic Number"**, submitted by **Jowahira Tabassum**, Roll No. **MAT-19/23**, Registration No. **MSSV-0023-101-001376**, in partial fulfillment for the award of the degree of **Master of Science in Mathematics**, is a bonafide record of original work carried out under my supervision and guidance.

To the best of my knowledge, the work has not been submitted earlier to any other institution for the award of any degree or diploma.

Dr. Raju Bordoloi
Head Of The Department
Department of Mathematics
Mahapurusha Srimanta Sankaradeva Viswavidyalaya

Date:
Place:

Signature of Guide

Declaration

I, **Jowahira Tabassum**, hereby declare that the dissertation titled **"An Introduction To The p-Adic Number"**, submitted to the Department of Mathematics, **Mahapurusha Srimanta Sankaradeva Viswavidyalaya**, is a record of original work carried out by me under the supervision of **Dr. Raju Bordoloi**, Head Of The Department.
This work has not been submitted earlier to any other institution or university for the award of any degree or diploma.

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CHAPTER 1

Introduction

The p-adic numbers, for any prime number p, originate from a different definition of the distance between two rational numbers. The Euclidean absolute value—the standard distance function-leads to the construction of the real numbers. Although real numbers are more familiar and intuitive to most, this paper aims to treat the p-adic numbers on equal footing. For instance, both the real numbers $\mathbb{R}$ and the p-adic numbers $\mathbb{Q}_p$ are complete metric spaces.

While the real numbers form a standard metric space, the p-adic numbers form an *ultrametric* space, resulting in many fascinating but often counterintuitive conclusions. Despite this, the p-adic numbers are computationally convenient and, in some contexts, more effective than real numbers. They find applications across number theory, analysis, algebra, and other mathematical domains.

One notable application is **Hensel's Lemma**, which is used for finding roots of polynomials. Another is **Mahler's Theorem**, a p-adic analogue of the classical Stone–Weierstrass Theorem. Yet another is **Monsky's Theorem**, concerning the impossibility of dissecting a square into an odd number of triangles of equal area—a result that relies crucially on 2-adic numbers.

Although the reader is encouraged to explore such theorems independently, this paper is focused on presenting a broad introduction to the p-adic numbers. We begin by defining the p-adic metric, and then use it to construct the field of p-adic numbers via completion of $\mathbb{Q}$ with respect to Cauchy sequences.

We then explore the distinctive behavior of sequences and series in the p-adic setting, which sharply contrasts with that in real analysis. This naturally leads us to the concept of p-adic expansions, offering a concrete way to represent and compute with p-adic numbers.

Next, we study the topology of $\mathbb{Q}_p$, which will help us understand the structure of p-adic spaces. Finally, we examine how p-adic geometry differs from its Euclidean counterpart, uncovering both theoretical and practical implications of this alternate number system.

CHAPTER 2

The p-adic Metric

Introducing a general definition of absolute value for an arbitrary field K. An *absolute value* (equivalently called a *valuation* or *norm*) is a function that attributes a non-negative real number to every element of K, in such a manner as to extend the usual absolute value on $\mathbb{R}$ or $\mathbb{Q}$. This function helps us to quantify the "size" or "magnitude" of elements in K, and it plays a critical role in building various kinds of metrics (distance functions) on the field. Specifically, in the investigation of *p-adic numbers*, we are concerned with a particular kind of absolute value known as the *p-adic absolute value*, so we have to define the *p-adic metric*—a non-Archimedean metric that is different from the well-known Euclidean metric.

Definition 2.1: Absolute Value on a Field

Let $\mathbb{K}$ be a field. An **absolute value** on $\mathbb{K}$ is a function

$$|\cdot| : \mathbb{K} \to \mathbb{R}_{\geq 0}$$

which assigns to each element $x \in \mathbb{K}$ a non-negative real number $|x|$, and satisfies the following properties for all $x, y \in \mathbb{K}$:

1. **Zero Property (Non-negativity):**

$$|x| = 0 \iff x = 0$$

2. **Multiplicativity:**

$$|xy| = |x||y|$$

3. **Triangle Inequality:**

$$|x + y| \leq |x| + |y|$$

Strong Triangle Inequality (Non-Archimedean Property)

In some cases, a stronger version of the triangle inequality holds:

$$|x + y| \leq \max(|x|, |y|)$$

If an absolute value satisfies this inequality, it is called **non-Archimedean**.

Remark 2.2

If the strong triangle inequality holds, then the standard triangle inequality is automatically satisfied.
This follows from the fact that:

$$\max(|x|, |y|) \leq |x| + |y|$$

Therefore:

$$|x + y| \leq \max(|x|, |y|) \leq |x| + |y|$$

Hence, any non-Archimedean absolute value is also a valid absolute value under the usual definition.

Definition 2.3: Metric Derived from an Absolute Value

Given a field $\mathbb{K}$ with an absolute value $|\cdot|$, we define a metric $d : \mathbb{K} \times \mathbb{K} \to \mathbb{R}_{\geq 0}$ by:

$$d(x, y) = |x - y|$$

This function d satisfies the standard properties of a metric:

1. **Identity of Indiscernibles and Non-negativity:**

$$d(x, y) = 0 \iff x = y$$

2. **Symmetry:**

$$d(x, y) = d(y, x)$$

3. **Triangle Inequality:**

$$d(x, z) \leq d(x, y) + d(y, z)$$

If the absolute value is non-Archimedean (i.e., satisfies the strong triangle inequality), the resulting metric is called an **ultrametric**. Ultrametric spaces differ significantly from Euclidean spaces in their geometric properties.

Lemma 2.4: Ultrametric Characteristic

Let $\mathbb{K}$ be a field equipped with a non-Archimedean absolute value $|\cdot|$. Then for all $x, y \in \mathbb{K}$, if $|x| \neq |y|$, we have:

$$|x + y| = \max(|x|, |y|)$$

Proof (Sketch)

Let us assume without loss of generality that $|x| < |y|$. Then by the strong triangle inequality:

$$|x + y| \leq \max(|x|, |y|) = |y|$$

But since $|x| < |y|$, it follows that $|x + y| = |y| = \max(|x|, |y|)$.

Exercise 2.5: Verifying Metric Properties

Let $d(x, y) = |x - y|$, where $|\cdot|$ is an absolute value on a field $\mathbb{K}$. Prove that d satisfies the following metric space axioms:

1. **Identity of Indiscernibles:**

$$d(x, y) = 0 \iff x = y$$

2. **Symmetry:**

$$d(x, y) = d(y, x)$$

3. **Triangle Inequality:**

$$d(x, z) \leq d(x, y) + d(y, z)$$

These properties follow directly from the definition of an absolute value.

Definition 2.6: p-adic Valuation

Let p be a fixed prime number. Then **p-adic valuation** v_p of a rational number measures the exponent of p in its prime factorization.

- For a non-zero integer a, define $v_p(a)$ as the largest integer n such that:

$$p^n \mid a$$

- For a non-zero rational number $\frac{a}{b}$ in lowest terms, define:

$$v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b)$$

- Define:

$$v_p(0) = +\infty$$

Using this valuation, the **p-adic absolute value** is defined as:

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This defines a non-Archimedean absolute value, and the associated metric $d_p(x, y) = |x - y|_p$ defines the **p-adic metric** on $\mathbb{Q}$.

Remark 2.7

The *p-adic valuation* $v_p(x)$ can be portray as a way to describe how divisible a number is by a prime p. In other words, it focuses on tracking of how many times a number can be divided by p before it can no longer be divided. For example,

$$v_2(40) = 3, \quad \text{since } 40 = 2^3 \cdot 5$$

To maintain consistency and to reflect that zero is divisible by every power of p, we define:

$$v_p(0) := +\infty$$

Exercise 2.8

Claim: The representation of a rational number has no bearing on the p-adic valuation. Specifically, if

$$x = \frac{a}{b} = \frac{a'}{b'} \in \mathbb{Q},$$

then

$$v_p(x) = v_p(a) - v_p(b) = v_p(a') - v_p(b').$$

Proof: Since $\frac{a}{b} = \frac{a'}{b'}$, we have

$$ab' = a'b.$$

Applying the p-adic valuation to both sides and using the multiplicative property, we get:

$$v_p(ab') = v_p(a'b) \Rightarrow v_p(a) + v_p(b') = v_p(a') + v_p(b).$$

Rearranging the terms:

$$v_p(a) - v_p(b) = v_p(a') - v_p(b').$$

Hence, the valuation $v_p(x)$ is independent of the way x is written as a fraction.

Exercise 2.9: Rational Numbers in Standard Form

Statement: Show that for any non-zero rational number $x \in \mathbb{Q}$, we can write

$$x = p^{v_p(x)} \cdot \frac{a}{b}, \quad \text{where } p \nmid a,\ p \nmid b.$$

Proof: Let $x = \frac{r}{s}$ be a rational number with $r, s \in \mathbb{Z}$, and $s \neq 0$.
Define:

$$v_p(r) = m, \quad v_p(s) = n.$$

Then:

$$v_p(x) = v_p\left(\frac{r}{s}\right) = m - n.$$

By definition of v_p, we can write:

$$r = p^m \cdot a, \quad \text{where } p \nmid a,$$

$$s = p^n \cdot b, \quad \text{where } p \nmid b.$$

Thus:

$$x = \frac{r}{s} = \frac{p^m \cdot a}{p^n \cdot b} = p^{m-n} \cdot \frac{a}{b} = p^{v_p(x)} \cdot \frac{a}{b}.$$

Conclusion: This states that any non-zero rational number x can be uniquely written as a product of a power of p and a fraction $\frac{a}{b}$ with $p \nmid a$ and $p \nmid b$, as required.

Lemma 2.10: Properties of the p-adic Valuation

Let $x, y \in \mathbb{Q}$. The p-adic valuation satisfies the following:

(i) Multiplicativity

$$v_p(xy) = v_p(x) + v_p(y)$$

Proof:

Assume that
$$x = \frac{a}{b}, \quad y = \frac{c}{d},$$
where $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$.
Then:
$$xy = \frac{ac}{bd}$$
Now apply the p-adic valuation:
$$v_p(xy) = v_p\left(\frac{ac}{bd}\right) = v_p(ac) - v_p(bd)$$
Using the multiplicative property of v_p on integers:
$$v_p(ac) = v_p(a) + v_p(c), \quad v_p(bd) = v_p(b) + v_p(d)$$
Therefore:
$$v_p(xy) = [v_p(a) + v_p(c)] - [v_p(b) + v_p(d)] = [v_p(a) - v_p(b)] + [v_p(c) - v_p(d)]$$
$$\Rightarrow v_p(xy) = v_p(x) + v_p(y)$$

Lemma 2.10 (ii): Non-Archimedean Inequality

Statement: For all $x, y \in \mathbb{Q}$,
$$v_p(x + y) \geq \min\left(v_p(x), v_p(y)\right)$$

Proof: Assume:
$$x = \frac{a}{b}, \quad y = \frac{c}{d}, \quad \text{with } a, b, c, d \in \mathbb{Z}, \quad b, d \neq 0$$
Then:
$$x + y = \frac{ad + bc}{bd} \Rightarrow v_p(x + y) = v_p(ad + bc) - v_p(bd)$$
We also know:
$$v_p(x) = v_p(a) - v_p(b), \quad v_p(y) = v_p(c) - v_p(d)$$
Let us apply the fundamental inequality of valuations:
$$v_p(r + s) \geq \min\left(v_p(r), v_p(s)\right) \quad (\text{for any } r, s \in \mathbb{Q})$$

Apply it to:

$$r = ad, \quad s = bc \Rightarrow v_p(ad + bc) \geq \min\left(v_p(ad), v_p(bc)\right)$$

Now compute:

$$v_p(ad) = v_p(a) + v_p(d), \quad v_p(bc) = v_p(b) + v_p(c)$$

So:

$$v_p(x + y) = v_p(ad + bc) - v_p(bd) \geq \min\left(v_p(a) + v_p(d),\ v_p(b) + v_p(c)\right) - [v_p(b) + v_p(d)]$$

Simplifying:

$$v_p(x + y) \geq \min\left(v_p(a) - v_p(b),\ v_p(c) - v_p(d)\right) = \min\left(v_p(x), v_p(y)\right)$$

Definition 2.11: The p-adic Absolute Value

On the field of rational numbers $\mathbb{Q}$, we define a new kind of absolute value called the **p-adic absolute value**, for any prime number p, as follows:

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Here, $v_p(x)$ denotes the *p-adic valuation* of x, which quantifies how divisible x is by the prime p.

- If $x = 0$, then $|x|_p = 0$. - If $x \neq 0$, then $v_p(x) \in \mathbb{Z}$, so $p^{-v_p(x)} \in \mathbb{R}_{>0}$.
This function measures a number's "p-divisibility": the more divisible x is by p, the smaller the value of $|x|_p$.

Example:

Let

$$x = \frac{75}{28}, \quad p = 5$$

We factor:

$$75 = 3 \cdot 5^2, \quad \Rightarrow v_5(75) = 2$$

$$28 = 2^2 \cdot 7, \quad \Rightarrow v_5(28) = 0$$

Therefore:

$$v_5\left(\frac{75}{28}\right) = v_5(75) - v_5(28) = 2 - 0 = 2$$

$$\Rightarrow \left|\frac{75}{28}\right|_5 = 5^{-2} = \frac{1}{25}$$

Remark 2.12: Value Discreteness

The **p-adic absolute value** takes values in a discrete set, because the p-adic valuation $v_p(x)$ is always an integer for any nonzero rational number $x \in \mathbb{Q} \setminus \{0\}$.

Hence, the image of the p-adic absolute value function is:

$$|x|_p \in \{p^n \mid n \in \mathbb{Z}\} \cup \{0\}$$

This means that $|x|_p$ is not dense in $\mathbb{R}_{\geq 0}$, unlike the usual absolute value $|\cdot|$, which maps $\mathbb{Q}$ onto a dense subset of the non-negative reals.

For example, possible values of $|x|_p$ include:

$$1,\ \frac{1}{p},\ \frac{1}{p^2},\ p,\ p^2,\ \ldots$$

These values jump discretely rather than vary continuously. This discrete nature of $|x|_p$ plays a fundamental role in the unique topology of the p-adic numbers.

Proposition 2.13: The p-adic Absolute Value is Non-Archimedean

We demonstrate that the p-adic absolute value satisfies the axioms of an absolute value and is non-Archimedean.

Step 1: Identity Axiom

We must show that:

$$|x|_p = 0 \iff x = 0$$

Proof:
By definition of the p-adic absolute value:

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

So if $x = 0$, clearly $|x|_p = 0$.
Conversely, if $x \neq 0$, then $v_p(x) \in \mathbb{Z}$, and:

$$|x|_p = p^{-v_p(x)} > 0$$

Hence:

$$|x|_p = 0 \iff x = 0$$

Step 2 : Multiplicativity

Let us show that the p-adic absolute value satisfies the multiplicative property:

$$|xy|_p = |x|_p \cdot |y|_p$$

Case 1: Suppose $x \neq 0$ and $y \neq 0$. Then:

$$v_p(xy) = v_p(x) + v_p(y)$$

BY Applying the definition of the p-adic absolute value:

$$|xy|_p = p^{-v_p(xy)} = p^{-(v_p(x)+v_p(y))} = p^{-v_p(x)} \cdot p^{-v_p(y)} = |x|_p \cdot |y|_p$$

Case 2: If either $x = 0$ or $y = 0$, then:

$$|xy|_p = |0|_p = 0, \quad \text{and} \quad |x|_p \cdot |y|_p = 0 \cdot |y|_p = 0$$

Conclusion: Also true for zero

Step 3 : Non-Archimedean (Ultrametric) Inequality

We now prove that the p-adic absolute value satisfies the ultrametric inequality:

$$|x + y|_p \leq \max\left(|x|_p, |y|_p\right)$$

Case: Suppose $x + y \neq 0$. If $x + y = 0$, the inequality holds trivially since $|x + y|_p = 0 \leq \max(|x|_p, |y|_p)$.

Without loss of generality, assume:

$$|x|_p \geq |y|_p \quad \Rightarrow \quad v_p(x) \leq v_p(y)$$

From Lemma 2.10(ii), we know:

$$v_p(x + y) \geq \min\left(v_p(x), v_p(y)\right) = v_p(x) \Rightarrow -v_p(x + y) \leq -v_p(x)$$

Exponentiating with base p, we get:

$$p^{-v_p(x+y)} \leq p^{-v_p(x)} \Rightarrow |x+y|_p \leq |x|_p = \max(|x|_p, |y|_p)$$

Conclusion: The ultrametric inequality holds:

$$|x+y|_p \leq \max(|x|_p, |y|_p)$$

Remark: This strong form of the triangle inequality contrasts with the classical absolute value, which only satisfies:

$$|x+y| \leq |x| + |y|$$

The stronger inequality is what makes the p-adic absolute value *non-Archimedean*.

Step 4: Strong Triangle Inequality and Metric Properties

Let $d(x,y) := |x-y|_p$. We verify that this function satisfies all the axioms of a metric on $\mathbb{Q}$.

Axiom 1: Non-negativity

$$|x-y|_p \geq 0 \quad \text{for all } x, y \in \mathbb{Q}$$

This follows directly from the definition of $|\cdot|_p$.

Axiom 2: Identity of Indiscernibles

$$|x-y|_p = 0 \iff x = y$$

This is a direct consequence of the identity property of the p-adic absolute value.

Axiom 3: Symmetry

$$|x-y|_p = |y-x|_p$$

Follows from the fact that $v_p(x-y) = v_p(y-x)$, hence their absolute values are equal.

Axiom 4: Strong Triangle Inequality (Ultrametric Property)

$$|x-z|_p \leq \max\left(|x-y|_p, |y-z|_p\right)$$

This holds because of the ultrametric inequality proven earlier in Step 3.

i.e,

All metric axioms are satisfied by the function:

$$d(x, y) := |x - y|_p$$

Therefore, it defines a **non-Archimedean metric** on $\mathbb{Q}$.

Conclusion

In conclusion, the p-adic absolute value:

- Satisfies all four axioms of an absolute value.
- Obeys the strong triangle inequality:

  $$|x + y|_p \leq \max(|x|_p, |y|_p)$$

  which classifies it as **non-Archimedean**.
- Induces a topology on $\mathbb{Q}$ that is completely different from the usual Euclidean topology.

Using this absolute value, we define the p-adic metric:

$$d_p(x, y) := |x - y|_p$$

This metric makes $(\mathbb{Q}, d_p)$ a metric space. By taking the completion of $\mathbb{Q}$ with respect to this metric, we obtain:

$$\mathbb{Q}_p := \text{completion of } \mathbb{Q} \text{ under } d_p$$

Thus, the field of p-adic numbers $\mathbb{Q}_p$ is constructed via completion of the rationals under the p-adic metric.

Exercise 2.14

Determine the following values of p-adic absolute values and metric:

1. $\left|\dfrac{75}{73}\right|_5$
2. $\left|\dfrac{16}{81}\right|_3$
3. $d_{11}(89, 4082)$, where $d_p(x, y) := |x - y|_p$

Detailed Instructions for $\left|\frac{75}{73}\right|_5$

We aim to compute the 5-adic absolute value of the rational number $\frac{75}{73}$.

Step 1: Use the Definition

Recall the definition of the p-adic absolute value:

$$|x|_p = p^{-v_p(x)}$$

Therefore,

$$\left|\frac{75}{73}\right|_5 = 5^{-v_5\left(\frac{75}{73}\right)} = 5^{-v_5(75)+v_5(73)}$$

Step 2: Prime Factorizations

We factor both the numerator and the denominator:

$$75 = 3 \cdot 5^2 \Rightarrow v_5(75) = 2$$

$$73 \text{ is a prime number not divisible by } 5 \Rightarrow v_5(73) = 0$$

Step 3: Substitute the Values

$$v_5\left(\frac{75}{73}\right) = v_5(75) - v_5(73) = 2 - 0 = 2$$

$$\left|\frac{75}{73}\right|_5 = 5^{-2} = \frac{1}{25}$$

$$\left|\frac{75}{73}\right|_5 = \frac{1}{25}$$

Detailed Instructions for $\left|\frac{16}{81}\right|_3$

We now compute the 3-adic absolute value of the rational number $\frac{16}{81}$.

Step 1: Prime Factorization

- $16 = 2^4 \Rightarrow v_3(16) = 0$
- $81 = 3^4 \Rightarrow v_3(81) = 4$

Step 2: Compute the p-adic Valuation

$$v_3\left(\frac{16}{81}\right) = v_3(16) - v_3(81) = 0 - 4 = -4$$

Step 3: Apply the Definition of p-adic Absolute Value

$$\left|\frac{16}{81}\right|_3 = 3^{-v_3\left(\frac{16}{81}\right)} = 3^{-(-4)} = 3^4 = 81$$

$$\left|\frac{16}{81}\right|_3 = 81$$

Detailed Instructions for $d_{11}(89, 4082)$

We are asked to compute the 11-adic distance between 89 and 4082.

Step 1: Use the Definition of the p-adic Metric

The p-adic distance is defined by:

$$d_p(x, y) = |x - y|_p$$

In this case, $p = 11$, $x = 89$, and $y = 4082$. So:

$$x - y = 89 - 4082 = -3993$$

Step 2: Factor 3993 to Find $v_{11}(3993)$

We determine how many times 11 divides 3993:

$$\begin{aligned}
3993 \div 11 = 363 \quad &\Rightarrow 11 \mid 3993 \\
363 \div 11 = 33 \quad &\Rightarrow 11^2 \mid 3993 \\
33 \div 11 = 3 \quad &\Rightarrow 11^3 \mid 3993 \\
3 \not\equiv 0 \mod 11 \quad &\Rightarrow 11^4 \nmid 3993
\end{aligned}$$

Therefore, $v_{11}(3993) = 3$

Step 3: Apply the Definition of the Absolute Value

$$|x - y|_{11} = |-3993|_{11} = 11^{-v_{11}(3993)} = 11^{-3} = \frac{1}{1331}$$

$$d_{11}(89, 4082) = \frac{1}{1331}$$

Exercise 2.15: Show that $\lim_{n\to\infty} p^n = 0$ in the p-adic metric

At first glance, this may seem counterintuitive, since in the usual (real) metric, $p^n \to \infty$. However, in the p-adic metric, distance is measured in terms of divisibility by p, not size.

Solution:

We aim to prove:

$$\lim_{n\to\infty} p^n = 0 \quad \text{in the } p\text{-adic metric.}$$

Step 1: Set Up the ε-Definition of a Limit

Let $\varepsilon > 0$ be given. We want to find $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|p^n|_p < \varepsilon.$$

Step 2: Recall the p-adic Absolute Value

We know that:

$$v_p(p^n) = n \quad \Rightarrow \quad |p^n|_p = p^{-n}.$$

Step 3: Use the Archimedean Property

Since the real numbers are Archimedean, there exists $N \in \mathbb{N}$ such that:

$$\frac{1}{p^N} < \varepsilon.$$

Then for all $n \geq N$, we have:

$$|p^n|_p = p^{-n} \leq p^{-N} < \varepsilon.$$

Hence, by the definition of a limit in a metric space:

$$\lim_{n\to\infty} p^n = 0 \quad \text{in the } p\text{-adic metric.}$$

This demonstrates that higher powers of p get *closer and closer to zero* in the p-adic world.

Theorem 2.16 (Ostrowski's Theorem)

Statement:
Any nontrivial absolute value on $\mathbb{Q}$ is equivalent to either the standard absolute value or a p-adic absolute value for some prime p.

Explanation:

This theorem provides a classification of all possible nontrivial absolute values on the field of rational numbers $\mathbb{Q}$. It states that:

- If an absolute value $|\cdot|$ on $\mathbb{Q}$ is **nontrivial** (i.e., it is not the absolute value where $|x| = 1$ for all $x \neq 0$),
- and if $|\cdot|$ satisfies the absolute value axioms:
    1. $|x| \geq 0$, with equality if and only if $x = 0$
    2. $|xy| = |x||y|$
    3. $|x + y| \leq |x| + |y|$ (triangle inequality)

Then the absolute value is **equivalent** to either:

1. The usual (Archimedean) absolute value on $\mathbb{Q}$:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

2. A p-adic absolute value $|\cdot|_p$ for some prime p, defined by:

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

No other fundamentally different absolute values exist on $\mathbb{Q}$ apart from these two types. Ostrowski's Theorem shows that all nontrivial absolute values on $\mathbb{Q}$ are already known and understood.

Significance of Ostrowski's Theorem

Ostrowski's Theorem is fundamental because it tells us the following:

- **No new type of absolute value on $\mathbb{Q}$ can be invented.** Every nontrivial absolute value on $\mathbb{Q}$ is equivalent to either:

1. the *usual* (Euclidean) absolute value $|\cdot|$, or
2. a *p-adic* absolute value $|\cdot|_p$ for some prime p.

- This justifies the importance of studying p-adic absolute values — they are the only alternatives to the Euclidean norm when considering absolute values on $\mathbb{Q}$.
- The **real numbers** $\mathbb{R}$ arise as the *completion of* $\mathbb{Q}$ under the standard absolute value:
$$\mathbb{R} = \widehat{\mathbb{Q}} \text{ with respect to } |\cdot|.$$
- The **p-adic numbers** $\mathbb{Q}_p$ arise as the *completions of* $\mathbb{Q}$ under the p-adic absolute values:
$$\mathbb{Q}_p = \widehat{\mathbb{Q}} \text{ with respect to } |\cdot|_p.$$
- **Together,** $\mathbb{R}$ **and all the** $\mathbb{Q}_p$ (for all primes p) account for all possible completions of $\mathbb{Q}$ with respect to absolute values.

Conclusion: Ostrowski's Theorem not only classifies all absolute values on $\mathbb{Q}$, but also provides the groundwork for understanding the structure of $\mathbb{Q}$'s completions. It motivates the study of p-adic analysis as a parallel theory to real analysis.

CHAPTER 3

Formulation of $\mathbb{Q}_p$ and $\mathbb{Z}_p$

Just as the field of real numbers $\mathbb{R}$ is constructed by completing the rational numbers $\mathbb{Q}$ with respect to the Euclidean absolute value $|\cdot|$, we can construct the field of p-adic numbers $\mathbb{Q}_p$ by completing $\mathbb{Q}$ with respect to the p-adic absolute value $|\cdot|_p$.

In the case of real numbers:

- Define a metric on $\mathbb{Q}$ by $d(x, y) = |x - y|$.
- Form equivalence classes of Cauchy sequences to account for "gaps" in $\mathbb{Q}$.
- The resulting completion is $\mathbb{R}$.

Analogously, for p-adic numbers:

- Define a metric on $\mathbb{Q}$ by $d_p(x, y) = |x - y|_p$, where $|\cdot|_p$ is the p-adic absolute value.
- Cauchy sequences under this metric are used to "fill in" the structure of $\mathbb{Q}$.
- The completion is denoted $\mathbb{Q}_p$, the field of p-adic numbers.

The set of p-adic integers, denoted $\mathbb{Z}_p$, is the subring of $\mathbb{Q}_p$ consisting of all elements with p-adic absolute value less than or equal to 1:

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}.$$

In this way, we construct a new number system based on divisibility by p, analogous to $\mathbb{R}$, but distinct in topology and arithmetic structure.

Definition 3.1: Cauchy Sequence

A **Cauchy sequence** is a sequence in which, as it progresses, the terms become arbitrarily close to one another.

Formally, a sequence (a_n) is called a Cauchy sequence if:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall m, n \geq N, \quad |a_n - a_m| < \varepsilon.$$

Justification: This means that the "tail" of the sequence becomes tightly packed. That is, past a certain point, all terms in the sequence are close to one another, regardless of whether a limit is known or exists within the given space.

For example, a sequence in $\mathbb{Q}$ approximating $\sqrt{2}$ is Cauchy but does *not* converge in $\mathbb{Q}$, since $\sqrt{2} \notin \mathbb{Q}$.

Proposition 3.2: $\mathbb{Q}$ Is Not Complete Under the p-adic Metric

Statement: The set of rational numbers $\mathbb{Q}$, when equipped with the p-adic metric, is not complete.

Justification:
There exist Cauchy sequences in $\mathbb{Q}$ with respect to the p-adic metric that do *not* converge to any element of $\mathbb{Q}$.

This is similar to how sequences such as $1, 1.4, 1.41, 1.414, \ldots$ are Cauchy in $\mathbb{Q}$ with respect to the usual absolute value but do not converge in $\mathbb{Q}$, since their limit $\sqrt{2} \notin \mathbb{Q}$.

Interpretation:
The field $\mathbb{Q}_p$ of p-adic numbers is obtained by completing $\mathbb{Q}$ with respect to the p-adic metric. It contains all such "missing" limit points.

Proof Sketch:
Refer to standard texts such as *Koblitz, "p-adic Numbers, p-adic Analysis, and Zeta-Functions"*, pp. 63–64.
One commonly cited example of a p-adic Cauchy sequence that does not converge in $\mathbb{Q}$ is:

$$x_n = \sum_{k=0}^{n} p^k = \frac{p^{n+1} - 1}{p - 1}.$$

This sequence is Cauchy in the p-adic metric, but it does not converge in $\mathbb{Q}$; its limit lies in $\mathbb{Q}_p$.

Theorem 3.3: Completion of a Field with Absolute Value

Statement: Let K be a field equipped with a nontrivial absolute value $|\cdot|$. Then there exists a complete field K', along with an absolute value $|\cdot|'$, such that:

1. $|\cdot|'$ extends $|\cdot|$,
2. K is dense in K',
3. The pair $(K', |\cdot|')$ is unique up to isomorphism.

Proof Sketch:
Step 1: Define the Set of Cauchy Sequences in K Let

$$\mathcal{C} := \left\{ (a_n) \in K^{\mathbb{N}} \mid (a_n) \text{ is Cauchy with respect to } |\cdot| \right\}.$$

Then $\mathcal{C}$ is a commutative ring under pointwise addition and multiplication.

Step 2: Define an Equivalence Relation on Cauchy Sequences We say:

$$(a_n) \sim (b_n) \iff \lim_{n\to\infty} |a_n - b_n| = 0.$$

This defines an equivalence relation. Denote the set of equivalence classes by $K' := \mathcal{C}/\sim$, and for each Cauchy sequence (a_n), let the corresponding class be $[(a_n)] \in K'$.

Step 3: Define Addition and Multiplication on K'

$$[(a_n)] + [(b_n)] := [(a_n + b_n)], \quad [(a_n)] \cdot [(b_n)] := [(a_n \cdot b_n)].$$

These operations are well-defined due to the triangle inequality and the stability of limits.

Step 4: Define the Extended Absolute Value $|\cdot|'$ For each equivalence class $[(a_n)] \in K'$, define:

$$|[(a_n)]|' := \lim_{n\to\infty} |a_n|.$$

This limit exists and is independent of the representative of the class.

Step 5: Show K' is a Field The additive identity is $[(0, 0, 0, \dots)]$, and the multiplicative identity is $[(1, 1, 1, \dots)]$. If $[(a_n)] \neq 0$, then for large n, $a_n \neq 0$, so (a_n^{-1}) is a Cauchy sequence and $[(a_n^{-1})]$ is the multiplicative inverse.

Step 6: Embed K Densely into K' Define the embedding:

$$\varphi : K \hookrightarrow K', \quad a \mapsto [(a, a, a, \dots)].$$

This is an injective ring homomorphism and its image is dense in K'.

Step 7: Prove K' is Complete Let $[(a_n^{(m)})]$ be a Cauchy sequence in K'. Define the diagonal sequence to extract a limit:

$$b_n := a_n^{(n)}.$$

Then $[(b_n)] \in K'$ is the limit, ensuring completeness.

Step 8: Prove Uniqueness Up to Isomorphism If K_1' and K_2' are two completions of K, then there exists a unique isomorphism:

$$\phi : K_1' \to K_2'$$

that fixes every element of K and preserves the extended absolute value.

Therefore, We have constructed a complete field K' with absolute value $|\cdot|'$ such that:

$$K \hookrightarrow K', \quad |\cdot|' \text{ extends } |\cdot|, \quad \text{and } K \text{ is dense in } K',$$

and this construction is unique up to isomorphism.

Definition 3.4: The p-adic Number Field $\mathbb{Q}_p$

The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the rational numbers $\mathbb{Q}$ with respect to the p-adic metric.
The p-adic absolute value is defined for any nonzero rational number x as:

$$|x|_p = p^{-v_p(x)}, \quad \text{where } v_p(x) \in \mathbb{Z}$$

and $v_p(x)$ denotes the exponent of p in the prime factorization of x. For $x = 0$, we set $|0|_p = 0$.
To complete $\mathbb{Q}$ under this metric, we include all limits of Cauchy sequences that do not already lie in $\mathbb{Q}$. The resulting field $\mathbb{Q}_p$ is complete and contains $\mathbb{Q}$ as a dense subfield. The existence and uniqueness of this completion follow from Theorem 3.4.

Remark 3.5: Uniqueness of p-adic Finalizations

According to Ostrowski's Theorem, the only non-trivial absolute values on $\mathbb{Q}$ are:

- The usual (Euclidean) absolute value, which leads to $\mathbb{R}$,
- The p-adic absolute values $|\cdot|_p$ for each prime p, leading to $\mathbb{Q}_p$.

Thus, the only significant metric completions of $\mathbb{Q}$ are:

$$\mathbb{R} \quad \text{and} \quad \mathbb{Q}_p \text{ for each prime } p.$$

Definition 3.6: The Ring of p-adic Integers $\mathbb{Z}_p$

The ring of p-adic integers is defined as:

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}.$$

Equivalently, $\mathbb{Z}_p$ consists of all elements of $\mathbb{Q}_p$ with non-negative p-adic valuation. This forms a subring of $\mathbb{Q}_p$, closed under both addition and multiplication.

Remark 3.7: Inclusion of Integers in $\mathbb{Z}_p$

If $x \in \mathbb{Z}$ and $v_p(x) \geq 0$, then:

$$|x|_p = p^{-v_p(x)} \leq 1 \Rightarrow x \in \mathbb{Z}_p.$$

Hence, we have:

$$\mathbb{Z} \subseteq \mathbb{Z}_p \quad \text{and} \quad \mathbb{Q} \subseteq \mathbb{Q}_p.$$

Although $\mathbb{Z}$ is countable, $\mathbb{Z}_p$ is uncountable, containing limits of sequences from $\mathbb{Z}$ that converge in the p-adic topology but not in the Euclidean topology.

p-adic Completion of $\mathbb{Z}$: Consider the sequence:

$$x_n = 1 + p + p^2 + \cdots + p^n.$$

This is a Cauchy sequence in the p-adic metric because the differences between successive terms are divisible by increasingly higher powers of p. This sequence converges in $\mathbb{Z}_p$ but not in $\mathbb{Z}$, showing that $\mathbb{Z}$ is not complete under the p-adic metric. Thus, $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ in the p-adic topology.

CHAPTER 4

Series and Sequences in $\mathbb{Q}_p$

This section examines the behavior of sequences and series in the p-adic number system $\mathbb{Q}_p$. One of the most significant characteristics of $\mathbb{Q}_p$ is that it forms an **ultrametric space**, which satisfies a stronger version of the triangle inequality:

$$|x + y|_p \leq \max(|x|_p, |y|_p),$$

with equality whenever $|x|_p \neq |y|_p$. This stronger condition substantially impacts convergence behavior.

Theorem 4.1 (Cauchy Sequences in $\mathbb{Q}_p$)

Let (a_n) be a sequence in $\mathbb{Q}_p$. Then (a_n) is a Cauchy sequence if and only if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \quad |a_{n+1} - a_n|_p < \varepsilon.$$

This result shows that in $\mathbb{Q}_p$, Cauchy convergence can be tested by just examining the difference between successive terms—a simplification not valid in $\mathbb{R}$.

Proof:

($\Rightarrow$) Direction:
Assume (a_n) is a Cauchy sequence in $\mathbb{Q}_p$. Then by definition, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have:

$$|a_n - a_m|_p < \varepsilon.$$

In particular, for all $n \geq N$, we get:

$$|a_{n+1} - a_n|_p < \varepsilon.$$

($\Leftarrow$) Direction:
Assume that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$|a_{n+1} - a_n|_p < \varepsilon.$$

Let $m, n \geq N$ with $n > m$, and write:

$$a_n - a_m = (a_{m+1} - a_m) + (a_{m+2} - a_{m+1}) + \cdots + (a_n - a_{n-1}).$$

By applying the ultrametric inequality:

$$|a_n - a_m|_p \leq \max\{|a_{m+1} - a_m|_p, |a_{m+2} - a_{m+1}|_p, \ldots, |a_n - a_{n-1}|_p\} < \varepsilon.$$

Hence, for all $m, n \geq N$, we have $|a_n - a_m|_p < \varepsilon$, so (a_n) is a Cauchy sequence.

Significance of Theorem 4.1

This result reduces the general Cauchy condition to a local condition on successive terms by using the strong triangle inequality (ultrametric property).
Such simplification is not available in Euclidean space $\mathbb{R}$, where the triangle inequality is too weak to infer global convergence from local behavior.
In $\mathbb{Q}_p$, however, the power of the non-Archimedean triangle inequality allows us to treat series and convergence with more tractable local conditions.

Corollary 4.2: Convergence Criterion in $\mathbb{Q}_p$

Let (a_n) be a sequence in $\mathbb{Q}_p$. Then the infinite series

$$\sum_{n=0}^{\infty} a_n$$

converges if and only if

$$\lim_{n\to\infty} a_n = 0.$$

This gives a precise condition for convergence of infinite series in $\mathbb{Q}_p$. Unlike in $\mathbb{R}$, where $\lim a_n = 0$ is necessary but not sufficient, in $\mathbb{Q}_p$ it is both necessary and sufficient.

Proof: Let (p_n) be the sequence of partial sums, where

$$p_n = a_0 + a_1 + \cdots + a_n.$$

Assume the series $\sum_{n=0}^{\infty} a_n$ converges in $\mathbb{Q}_p$. Then (p_n) converges, hence it is Cauchy.
By Theorem 4.1, this implies:

$$|p_{n+1} - p_n|_p = |a_{n+1}|_p < \varepsilon$$

for sufficiently large n. Hence $\lim_{n\to\infty} a_n = 0$.
Now assume $\lim_{n\to\infty} a_n = 0$. Then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n|_p < \varepsilon$ for all $n \geq N$. It follows that:

$$|p_{n+1} - p_n|_p = |a_{n+1}|_p < \varepsilon$$

so (p_n) is Cauchy. Since $\mathbb{Q}_p$ is complete, (p_n) converges, and the series $\sum a_n$ converges.

Example 4.3: A p-adic Geometric Series

Consider the series

$$\sum_{n=0}^{\infty} p^n.$$

We claim that it converges in $\mathbb{Q}_p$ and the sum is:

$$\sum_{n=0}^{\infty} p^n = \frac{1}{1-p}.$$

Proof: Let the partial sums be:

$$a_n = \sum_{i=0}^{n} p^i.$$

Recall the identity for geometric series:

$$1 - p^{n+1} = (1-p)(1 + p + p^2 + \cdots + p^n) = (1-p)a_n,$$

so we can solve for a_n:

$$a_n = \frac{1 - p^{n+1}}{1-p}.$$

Taking limits in $\mathbb{Q}_p$, since $\lim_{n\to\infty} p^{n+1} = 0$, we get:

$$\lim_{n\to\infty} a_n = \frac{1}{1-p}.$$

Thus, the series converges and:

$$\sum_{n=0}^{\infty} p^n = \frac{1}{1-p}.$$

Remark 4.4: Geometric Series Formula in $\mathbb{Q}_p$

This result is the p-adic analog of the real geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{when } |x| < 1 \text{ in } \mathbb{R}.$$

In $\mathbb{Q}_p$, the same formula holds for any $x \in \mathbb{Q}_p$ such that:

$$|x|_p < 1.$$

Thus, the geometric series converges and satisfies:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{for } |x|_p < 1.$$

CHAPTER 5

p-adic Expansions

Our goal in this section is to gain a better understanding of how p-adic numbers can be expressed in series form, much like binary numbers (e.g., 101.01_2) or decimal numbers (e.g., $0.333\ldots$) have expansions. However, in the p-adic world, the direction of convergence and interpretation of series behave quite differently.

Example 5.1: A Curious Case in $\mathbb{Q}_2$

In the 2-adic numbers, for instance, the following infinite sum converges:

$$-1 = 1 + 2 + 4 + 8 + \cdots$$

This may seem counterintuitive from the perspective of real numbers. However, this convergence makes sense in the p-adic world, where the p-adic norm governs convergence rather than the usual Euclidean norm.

Proposition 5.2: convergent of p-adic series

Any infinite series of the form

$$\sum_{n=n_0}^{\infty} a_n p^n$$

converges in $\mathbb{Q}_p$, provided that:

- $a_n \in \{0, 1, \ldots, p-1\}$ (i.e., digits in base p),
- $n_0 \in \mathbb{Z}$ (i.e., the series may begin at any integer index).

This tells us that any p-adic number can be represented as a series expansion in powers of p, with bounded coefficients.
Evidence: From Example 2.15, we know:

$$\lim_{n\to\infty} p^n = 0 \quad \text{in } \mathbb{Q}_p$$

This reflects how larger powers of p become smaller and smaller in the p-adic norm. Also, since $0 \le a_n < p$ for all n, we obtain:

$$\lim_{n\to\infty} a_n p^n = 0$$

By Corollary 4.2, which states that a series in $\mathbb{Q}_p$ converges if its terms go to zero, the whole series converges.

Note: The starting index n_0 does not affect convergence; it only shifts the entire series without changing the tail behavior.

Definition 5.3: p-adic Expansion

Let $\alpha \in \mathbb{Q}_p$. A **p-adic expansion** of α is a representation of the form:

$$\alpha = \sum_{n=n_0}^{\infty} a_n p^n$$

Where:

- $n_0 \in \mathbb{Z} \cup \{\infty\}$,
- $a_{n_0} \neq 0$ (to avoid leading zeros and ensure uniqueness),
- $0 \leq a_n < p$ for all $n \geq n_0$.

This is analogous to expressing a real number in base-p, but with key differences:

- The coefficients a_n belong to $\mathbb{Z}$ and are bounded.
- The direction of the infinite sum is typically to the right (i.e., toward increasing n).

Remark 0.1 (Finite Case: Natural Numbers). If $\alpha \in \mathbb{N}$, then its p-adic expansion is just its standard base-p representation:

$$\alpha = a_0 + a_1 p + a_2 p^2 + \cdots + a_N p^N$$

for some $N \in \mathbb{N}$, with $a_n = 0$ for $n > N$. Hence, the p-adic framework extends the usual base-p expansion of natural numbers.

Propotition 5.4: Uniqueness of Expansion

Each p-adic number $\alpha \in \mathbb{Q}_p$ has a unique p-adic expansion:

$$\alpha = \sum_{n=n_0}^{\infty} a_n p^n$$

Two different sequences of coefficients $\{a_n\}$ cannot yield the same p-adic number.
Suppose $\alpha \in \mathbb{Q}$ and its p-adic expansion is:

$$\sum_{n=n_0}^{\infty} a_n p^n$$

Prove that:

$$v_p(\alpha) = n_0$$

Explanation: The starting index n_0 of the expansion gives the p-adic valuation $v_p(\alpha)$, which denotes the highest power of p dividing α.

Example 5.5:

Let $\alpha = \frac{3}{8} = \frac{3}{2^3}$. Then $v_2(\alpha) = -3$, and the 2-adic expansion of α begins at $n_0 = -3$. This gives us intuitive insight. In base-10, we write:

$$345 = 3 \cdot 10^2 + 4 \cdot 10^1 + 5 \cdot 10^0$$

Similarly, in a p-adic expansion, we write coefficients with increasing powers of p:

$$\cdots + a_{-2}p^{-2} + a_{-1}p^{-1} + a_0 + a_1 p + a_2 p^2 + \cdots$$

However, in $\mathbb{Q}_p$, the "tail" is infinite (positive powers), not the "head" (negative powers), since only finitely many negative-indexed terms may appear.

Examples 5.6: Examples of p-adic Expansion

[Example 1: -1 in $\mathbb{Q}_2$]

$$-1 = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \cdots$$

This series diverges in $\mathbb{R}$ but converges in $\mathbb{Q}_2$ because $2^n \to 0$ in the 2-adic norm.
[Example 2: 175 in $\mathbb{Q}_5$]

$$175 = 1 \cdot 5^3 + 2 \cdot 5^2 + 0 \cdot 5^1 + 0 \cdot 5^0 = 125 + 50 + 0 + 0$$

The 5-adic expansion is:

$$175 = \sum_{n=0}^{3} a_n 5^n = 0 + 0 \cdot 5 + 2 \cdot 5^2 + 1 \cdot 5^3$$

[Example 3: $\frac{194}{7}$ in $\mathbb{Q}_7$] Base-7 interpretation:

$$\frac{194}{7} = 36.5_7 = 3 \cdot 7^1 + 6 \cdot 7^0 + 5 \cdot 7^{-1} = 21 + 6 + \frac{5}{7}$$

Hence, base subscripts indicate the prime used for expansion.

Verification in $\mathbb{Q}_2$

To verify:

$$-1 = \sum_{n=0}^{\infty} 2^n$$

Observe:

$$1 + (-1) = 1 + \sum_{n=0}^{\infty} 2^n = \sum_{n=0}^{\infty} 2^n + 1$$

This sum converges to 0 in $\mathbb{Q}_2$ due to the vanishing nature of 2^n as $n \to \infty$.

Lemma 5.7: Truncating a p-adic Expansion

Let $\alpha \in \mathbb{Z}_p$ be given by the expansion:

$$\alpha = \sum_{n=n_0}^{\infty} a_n p^n$$

For any $k \in \mathbb{N}$,

$$\alpha \equiv \sum_{n=n_0}^{k-1} a_n p^n \pmod{p^k}$$

Proof. Split the sum:

$$\alpha = \sum_{n=n_0}^{k-1} a_n p^n + \sum_{n=k}^{\infty} a_n p^n$$

The second sum is divisible by p^k since $p^k \mid p^n$ for $n \geq k$, so:

$$\sum_{n=k}^{\infty} a_n p^n \equiv 0 \pmod{p^k}$$

Hence:

$$\alpha \equiv \sum_{n=n_0}^{k-1} a_n p^n \pmod{p^k}$$

□

Application: Modular Reduction in $\mathbb{Z}_p$ This lemma is essential for modular computations in $\mathbb{Z}_p$.

Suppose we wish to compute $\alpha \bmod p^5$. According to Lemma 5.7, we do not need the full p-adic expansion of α.

$$\alpha = \sum_{n=n_0}^{\infty} a_n p^n \quad \Rightarrow \quad \alpha \equiv \sum_{n=n_0}^{4} a_n p^n \pmod{p^5}$$

Therefore, **only the first five coefficients** (from $n = n_0$ up to $n = 4$) are required to determine $\alpha \mod p^5$.

Example 5.8: Determine the 5-adic Expansion of $\frac{4}{3}$

Statement:
Determine the 5-adic expansion of the rational number

$$\frac{4}{3} \in \mathbb{Q}_5.$$

Solution:
We aim to express $\frac{4}{3}$ as an infinite sum of the form:

$$\frac{4}{3} = \sum_{n=0}^{\infty} a_n 5^n, \quad \text{where } a_n \in \{0, 1, 2, 3, 4\}.$$

This is the definition of the 5-adic expansion of a number in $\mathbb{Z}_5$.

Step 1: p-adic Valuation

Given that 3 and 4 are both coprime to 5, we compute the 5-adic valuation:

$$v_5\left(\frac{4}{3}\right) = v_5(4) - v_5(3) = 0 - 0 = 0.$$

Thus,

$$\left|\frac{4}{3}\right|_5 = 5^{v_5(4/3)} = 5^0 = 1.$$

Consequently, $\frac{4}{3} \in \mathbb{Z}_5$, the ring of 5-adic integers, since its 5-adic absolute value is 1.

Step 2: Determining a_0

By Lemma 5.7, for any 5-adic integer $x \in \mathbb{Z}_5$, the constant term a_0 in its expansion satisfies:

$$x \equiv a_0 \pmod{5}.$$

Apply this to $x = \frac{4}{3}$:

$$\frac{4}{3} \equiv a_0 \pmod{5} \quad \Rightarrow \quad 4 \equiv 3a_0 \pmod{5}.$$

To isolate a_0, multiply both sides by the inverse of 3 modulo 5. Since $3^{-1} \equiv 2 \pmod{5}$, we have:

$$a_0 \equiv 2 \cdot 4 = 8 \equiv 3 \pmod{5}.$$

Therefore, $a_0 = 3$.

Step 3: Determining a_1

Now apply Lemma 5.8 at the modulus $25 = 5^2$ level:

$$\frac{4}{3} \equiv a_0 + 5a_1 \pmod{25} \Rightarrow \frac{4}{3} \equiv 3 + 5a_1 \pmod{25}.$$

Subtract 3 from both sides:

$$\frac{4}{3} - 3 = \frac{4-9}{3} = \frac{-5}{3} \equiv 5a_1 \pmod{25}.$$

Multiply both sides by 3:

$$-5 \equiv 15a_1 \pmod{75} \Rightarrow -5 \equiv 15a_1 \pmod{25}.$$

Add 25 to both sides to simplify:

$$20 \equiv 15a_1 \pmod{25}.$$

Divide both sides by 5:

$$3a_1 \equiv 4 \pmod{5}.$$

Now multiply both sides by $3^{-1} \equiv 2 \pmod{5}$:

$$a_1 \equiv 2 \cdot 4 = 8 \equiv 3 \pmod{5}.$$

Therefore, $a_1 = 3$.

Step 4: Determining a_2

We now solve modulo $125 = 5^3$:

$$\frac{4}{3} \equiv a_0 + 5a_1 + 25a_2 \pmod{125}.$$

We already have:

$$a_0 = 3, \quad a_1 = 3 \quad \Rightarrow \quad a_0 + 5a_1 = 3 + 5 \cdot 3 = 18.$$

Thus,

$$\frac{4}{3} \equiv 18 + 25a_2 \pmod{125}.$$

Subtracting 18 from both sides:

$$\frac{4}{3} - 18 = \frac{4 - 54}{3} = \frac{-50}{3} \equiv 25a_2 \pmod{125}.$$

Now multiply both sides by 3:

$$-50 \equiv 75a_2 \pmod{375}.$$

Divide both sides by 25:

$$-2 \equiv 3a_2 \pmod{5}.$$

Now solve:

$$3a_2 \equiv -2 \equiv 3 \pmod{5} \quad \Rightarrow \quad a_2 \equiv 1 \pmod{5}.$$

Therefore, $a_2 = 1$.

Step 5: Examining the Trend

Continuing the process further, we observe the following coefficients in the 5-adic expansion:

$$a_0 = 3, \quad a_1 = 3, \quad a_2 = 1, \quad a_3 = 3, \quad a_4 = 3, \quad a_5 = 1, \quad \ldots$$

From this, we see a clear pattern:

$$(a_n) = (3, 3, 1, 3, 3, 1, \ldots),$$

which repeats in a cycle of length 3:

$$3, 3, 1.$$

Conclusion: The 5-adic expansion of $\frac{4}{3}$ is ultimately periodic with repeating digits:

$$\frac{4}{3} = (3, 3, 1, 3, 3, 1, \ldots)_5.$$

Or, using a bar to denote the repeating block:

$$\frac{4}{3} = (3\overline{3\ 1})_5$$

Verification through Multiplication

We verify that the 5-adic expansion correctly represents the number $\frac{4}{3}$.

The expansion obtained was:

$$\frac{4}{3} = 3 \cdot 5^0 + 3 \cdot 5^1 + 1 \cdot 5^2 + 3 \cdot 5^3 + 3 \cdot 5^4 + 1 \cdot 5^5 + \cdots$$

Multiply both sides by 3:

$$4 = 3 \cdot \left(3 \cdot 5^0 + 3 \cdot 5^1 + 1 \cdot 5^2 + 3 \cdot 5^3 + 3 \cdot 5^4 + 1 \cdot 5^5 + \cdots\right)$$

$$= 9 \cdot 5^0 + 9 \cdot 5^1 + 3 \cdot 5^2 + 9 \cdot 5^3 + 9 \cdot 5^4 + 3 \cdot 5^5 + \cdots$$

Since $|5^n|_5 \to 0$ as $n \to \infty$, the terms involving higher powers of 5 become arbitrarily small in the 5-adic norm. Thus, this infinite sum converges in $\mathbb{Q}_5$.
Hence, the product converges to:

$$\frac{4}{3} \cdot 3 = 4$$

Therefore, the expansion is verified.

Theorem 5.9: Rational p-adics Have Eventually Periodic Expansions

Statement:
A p-adic number has an eventually periodic expansion if and only if it is rational.

Proof:

Let $x \in \mathbb{Q}_p$.

($\Rightarrow$) Direction: Assume $x \in \mathbb{Q}$. We aim to show that its p-adic expansion eventually becomes periodic.
Write:

$$x = \sum_{n=0}^{\infty} a_n p^n,$$

where $a_n \in \{0, 1, \ldots, p-1\}$.
Since $x \in \mathbb{Q}$, when we reduce $x \mod p^k$ for increasing values of k, we are effectively tracking the sequence of remainders in a base-p system.
Because there are only finitely many congruence classes modulo p^k, and since the digits a_n are determined from these reductions, the sequence (a_n) must eventually repeat.
This is a direct consequence of the **Pigeonhole Principle**.
Hence, the expansion becomes periodic.

($\Leftarrow$) Direction: Assume the p-adic expansion of $x \in \mathbb{Q}_p$ is eventually periodic. Then we can write:

$$x = a_0 + a_1 p + \cdots + a_r p^r + (b_0 + b_1 p + \cdots + b_s p^s)\left(1 + p^t + p^{2t} + \cdots\right),$$

where the second part is the repeating portion.
The expression in parentheses is a geometric series:

$$\sum_{k=0}^{\infty} p^{kt} = \frac{1}{1-p^t}, \quad \text{since } |p^t|_p < 1.$$

Hence,

$$x = \text{finite sum} + (\text{rational term}) \cdot \frac{1}{1-p^t}.$$

Since all coefficients are rational, and the geometric series converges in $\mathbb{Q}_p$, it follows that $x \in \mathbb{Q}$.

Conclusion:

$$x \in \mathbb{Q} \iff \text{the } p\text{-adic expansion of } x \text{ is eventually periodic.}$$

CHAPTER 6

The Topology on $\mathbb{Q}_p$

The field of p-adic numbers, $\mathbb{Q}_p$, inherits not only an algebraic structure but also a topological one via the p-adic metric.

The p-adic Metric

Define the distance function:

$$d_p(x, y) := |x - y|_p,$$

where $|\cdot|_p$ is the p-adic absolute value.
This metric gives rise to a topology on $\mathbb{Q}_p$ that allows us to explore concepts such as convergence, continuity, open and closed sets, and compactness—within a framework quite different from the real number topology.

Comparison with the Real Numbers

- **Topology of $\mathbb{R}$:**
  - Dense and connected.
  - Ordered.
  - No "gaps" — the real line is complete in the traditional sense.
- **Topology of $\mathbb{Q}_p$:**
  - Totally disconnected.
  - Unordered.
  - Possesses a hierarchical (ultrametric) structure that resembles a fractal.

Understanding this topological distinction is essential when studying convergence and analysis in $\mathbb{Q}_p$, as the behavior of sequences and functions differs radically from the familiar setting of $\mathbb{R}$.

Theorem 6.1: Open Balls are Closed, and Closed Balls are Open in $\mathbb{Q}_p$

Statement:
Any open ball in $\mathbb{Q}_p$ is also a closed set, and vice versa.

Proof:

Let $B(a, r) := \{x \in \mathbb{Q}_p \mid |x - a|_p < r\}$ denote the open ball of radius $r > 0$ centered at $a \in \mathbb{Q}_p$.
Due to the nature of the p-adic absolute value, only discrete norms of the form p^n exist, and no values fall between two consecutive such powers.
Let $r > 0$ be given. Then we can choose the smallest integer n such that

$$r \leq p^{-n}.$$

Because there are no values of the norm between p^{-n} and p^{-n+1}, the condition $|x - a|_p < r$ becomes:

$$|x - a|_p < r \iff |x - a|_p \leq p^{-n} \iff x \in \overline{B}(a, r).$$

Thus, the open ball coincides with a closed ball:

$$B(a, r) = \{x \in \mathbb{Q}_p \mid |x - a|_p \leq p^{-n}\}.$$

Conversely, by symmetry of the metric ($d(x, y) = d(y, x)$), every closed ball is also open.
Conclusion: All open balls in $\mathbb{Q}_p$ are closed, and all closed balls are open. In other words, they are *clopen*.

Corollary 6.2: $\mathbb{Z}_p$ is Clopen in $\mathbb{Q}_p$

Justification:
By definition,

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\} = B(0, 1),$$

which is the closed ball of radius 1 centered at 0.
By Theorem 6.1, this ball is also open.
Therefore, $\mathbb{Z}_p$ is both open and closed (clopen) in $\mathbb{Q}_p$.
This clopen property implies that $\mathbb{Q}_p$ is not connected.

Corollary 6.3: $\mathbb{Q}_p$ is Totally Disconnected

Definition:

A topological space is *totally disconnected* if its only connected subsets are singletons.

Proof:

Let $a, b \in \mathbb{Q}_p$ with $a \neq b$. Define

$$\delta := |a - b|_p > 0.$$

Now consider the disjoint open balls:

$$A := \{x \in \mathbb{Q}_p \mid |x - a|_p < \delta/2\}, \quad B := \{x \in \mathbb{Q}_p \mid |x - b|_p < \delta/2\}.$$

Using the strong triangle inequality (ultrametric inequality), we know:

$$|x - a|_p < \delta/2 \text{ and } |x - b|_p < \delta/2 \Rightarrow |a - b|_p < \delta,$$

which contradicts our choice of $\delta = |a - b|_p$.
Therefore, $A \cap B = \emptyset$, but $A \cup B$ covers a neighborhood of both a and b. Hence, a and b lie in disjoint clopen sets.
Conclusion: Any two distinct points in $\mathbb{Q}_p$ can be separated by disjoint clopen sets, implying that $\mathbb{Q}_p$ is totally disconnected.

Theorem 6.4: The Ring $\mathbb{Z}_p$ is Compact

Goal: Show that $\mathbb{Z}_p$ is compact as a subset of the metric space $\mathbb{Q}_p$.
Key fact: In metric spaces, compactness is equivalent to sequential compactness.
Therefore, it is sufficient to show that every sequence in $\mathbb{Z}_p$ has a convergent subsequence whose limit is also in $\mathbb{Z}_p$.

Step 1: Consider a Sequence in $\mathbb{Z}_p$

Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{Z}_p$ be an arbitrary sequence. Each $a_n \in \mathbb{Z}_p$ admits a base-p expansion:

$$a_n = \sum_{i=0}^{\infty} a_{n,i} \cdot p^i, \quad \text{where } a_{n,i} \in \{0, 1, \ldots, p-1\}.$$

Step 2: Construct a Diagonal Subsequence

We will construct a subsequence (b_n) whose first k digits stabilize as $k \to \infty$.
Since $a_{n,0} \in \{0, 1, \ldots, p-1\}$, there exists some value b_0 such that $a_{n,0} = b_0$ for infinitely many n. Let $(a_n^{(0)})$ be the subsequence of (a_n) with 0th digit equal to b_0.
Repeat this argument for 1st digits: among $(a_n^{(0)})$, some digit b_1 appears infinitely often in the 1st place. Take a further subsequence $(a_n^{(1)})$ with $a_{n,1} = b_1$.

Continue inductively: for each $k \in \mathbb{N}$, select a subsequence $(a_n^{(k)}) \subset (a_n^{(k-1)})$ such that:

$$a_{n,i} = b_i \quad \text{for all } i = 0, 1, \ldots, k.$$

Now define the diagonal sequence:

$$b_n := a_n^{(n)}.$$

By construction, (b_n) converges in the p-adic metric to:

$$b := \sum_{i=0}^{\infty} b_i p^i \in \mathbb{Z}_p.$$

Conclusion: Every sequence in $\mathbb{Z}_p$ has a convergent subsequence with limit in $\mathbb{Z}_p$, so $\mathbb{Z}_p$ is sequentially compact. Hence:

$$\mathbb{Z}_p \text{ is compact.}$$

Corollary 6.5: The Field $\mathbb{Q}_p$ is Locally Compact

Goal: Show that every point in $\mathbb{Q}_p$ has a compact neighborhood.
Let $x \in \mathbb{Q}_p$. For some $k \in \mathbb{Z}$, define the closed ball of radius p^{-k}:

$$B := B(x, p^{-k}) = \{y \in \mathbb{Q}_p \mid |x - y|_p \leq p^{-k}\}.$$

This set can be written as:

$$B = x + p^k \mathbb{Z}_p,$$

which is a translation and dilation of $\mathbb{Z}_p$.
Since:

- $\mathbb{Z}_p$ is compact (Theorem 6.4),
- scalar multiplication and translation are continuous in $\mathbb{Q}_p$,

it follows that:

$$p^k \mathbb{Z}_p \text{ is compact,} \quad \text{so } x + p^k \mathbb{Z}_p \text{ is compact.}$$

Therefore: Every point $x \in \mathbb{Q}_p$ has a compact neighborhood $B(x, p^{-k})$.

$$\mathbb{Q}_p \text{ is locally compact.}$$

CHAPTER 7

Geometry in $\mathbb{Q}_p$

This section examines the behavior of geometric concepts, with a particular focus on triangles, in the field of p-adic numbers. These findings demonstrate the unexpected character of p-adic spaces and stand in stark contrast to well-known Euclidean geometry.

Theorem 7.1:

All triangles in $\mathbb{Q}_p$ are isosceles.
Let $a, b, c \in \mathbb{Q}_p$ be any three distinct points. Then the triangle with vertices a, b, c will always have at least two equal sides. That is, the triangle is always isosceles.

Proof:

Let $a, b, c \in \mathbb{Q}_p$ be three distinct points. The side lengths of the triangle $\triangle abc$ are defined using the p-adic metric as follows:

$$d_p(a,b) = |a-b|_p, \quad d_p(b,c) = |b-c|_p, \quad d_p(a,c) = |a-c|_p$$

We aim to show that at least two of these distances are equal.
Observe that:

$$a - c = (a-b) + (b-c)$$

Now, using the **ultrametric inequality** (non-Archimedean triangle inequality), we have:

$$|a-c|_p \leq \max\left(|a-b|_p, |b-c|_p\right)$$

From **Lemma 2.4**, we know the following stronger result: if $|a-b|_p \neq |b-c|_p$, then:

$$|a-c|_p = \max\left(|a-b|_p, |b-c|_p\right)$$

This implies that $|a-c|_p$ must be equal to either $|a-b|_p$ or $|b-c|_p$. In terms of distance:

$$d_p(a,c) = \max\left(d_p(a,b), d_p(b,c)\right)$$

So, either $d_p(a,c) = d_p(a,b)$ or $d_p(a,c) = d_p(b,c)$. Hence, **at least two sides of the triangle are equal in length**, and the triangle is **isosceles**.
Conclusion: Any triangle formed by three points in $\mathbb{Q}_p$ is necessarily isosceles.

Remark 7.2:

In $\mathbb{Q}_p$, the base of a triangle is the **shortest side**, provided the triangle is not equilateral.

Definition:

In standard Euclidean geometry, *any* side of a triangle can serve as the base. However, in $\mathbb{Q}_p$, the `maximum` function (as used in Lemma 2.4) determines which two sides are equal.
Assume the triangle has the following side lengths:

$$d_p(a, b) = r = d_p(b, c), \quad d_p(a, c) = s, \quad \text{with } s < r$$

Then:

- The two equal sides, r, are the longer sides.
- The remaining side s, which is strictly smaller than r, becomes the *base*.

This behavior arises from the non-Archimedean nature of the p-adic norm: the ultrametric inequality ensures that in a triangle, the longest two distances must be equal, and the third must be shorter.

Corollary 7.3: No Three Distinct Collinear Points in $\mathbb{Q}_p$

Statement: There do not exist three distinct points in $\mathbb{Q}_p$ that lie on the same straight line.

Proof. Assume, for contradiction, that three distinct points $a, b, c \in \mathbb{Q}_p$ are collinear.
In Euclidean geometry, collinearity of three points implies:

$$d_p(a, c) = d_p(a, b) + d_p(b, c)$$

However, the p-adic metric is non-Archimedean and satisfies the ultrametric inequality:

$$d_p(a, c) \leq \max\left(d_p(a, b), d_p(b, c)\right)$$

Thus, such additive behavior is not possible unless one of the terms is zero. But since a, b, c are distinct, all distances are positive.
Suppose the additive condition holds:

$$d_p(a, c) = d_p(a, b) + d_p(b, c)$$

Then $d_p(a, c) > d_p(a, b)$ and $d_p(a, c) > d_p(b, c)$, which makes $d_p(a, c)$ the longest side of triangle $\triangle abc$.
But by Theorem 7.1, any triangle in $\mathbb{Q}_p$ must be isosceles, meaning at least two side lengths must be equal.
Now, if the triangle satisfies a linear addition rule but has all side lengths distinct, it contradicts the ultrametric structure of $\mathbb{Q}_p$, which forbids such configurations.
Conclusion: A contradiction arises, and hence, there are no three distinct collinear points in $\mathbb{Q}_p$. □

Theorem 7.4: Maximum Number of Equally Spaced Points in $\mathbb{Q}_p$

Statement: At most p distinct points in $\mathbb{Q}_p$ can be equally spaced apart. That is, there cannot exist more than p points such that the distance between any pair of them is the same.

Proof. Assume, for contradiction, that there exist $p + 1$ distinct points $a_1, a_2, \ldots, a_{p+1} \in \mathbb{Q}_p$ such that for all $i \neq j$,

$$d_p(a_i, a_j) = r > 0$$

That is, all points are mutually equidistant.
From the definition of the p-adic norm, we know:

$$|a_i - a_j|_p = p^{-v_p(a_i - a_j)} = r \quad \text{(constant)}$$

Let $v_p(a_i - a_j) = m$ for all $i \neq j$. Then we can write:

$$a_i - a_j = p^m \cdot u_{ij}, \quad \text{where } u_{ij} \in \mathbb{Z}_p^\times$$

Thus, all differences $a_i - a_j$ are divisible by p^m but not by a higher power.
Now consider the p-adic expansions of the a_i's:

$$a_i = \sum_{n=n_0}^{\infty} a_{i,n} p^n, \quad \text{with } 0 \leq a_{i,n} < p$$

Define new elements:

$$a_i' := \sum_{n=0}^{\infty} a_{i,n+m} p^n \quad \Rightarrow \quad a_i = p^m \cdot a_i'$$

Then:

$$a_i - a_j = p^m(a'_i - a'_j) \Rightarrow v_p(a'_i - a'_j) = 0$$

This implies $a'_i - a'_j \in \mathbb{Z}_p^\times$, i.e., a unit with p-adic valuation 0. So the leading (zeroth) digit of a'_i and a'_j must differ.
Since each a'_i has the expansion:

$$a'_i = a_{i,m} + a_{i,m+1}p + a_{i,m+2}p^2 + \dots$$

and the digit $a_{i,m} \in \{0, 1, \dots, p-1\}$, there are only p possible values for the first digit. However, we assumed there are $p+1$ such elements. By the Pigeonhole Principle, two of them—say a'_i and a'_j—must have the same first digit:

$$a_{i,m} = a_{j,m} \Rightarrow a'_i \equiv a'_j \mod p \Rightarrow v_p(a'_i - a'_j) \geq 1$$

But this contradicts our earlier conclusion that $v_p(a'_i - a'_j) = 0$.
Contradiction. Therefore, the assumption that more than p such equidistant points exist must be false.
Conclusion: There can be at most p mutually equidistant points in $\mathbb{Q}_p$. □

Remark 7.5: Conceptual Commentary

This theorem highlights the surprising and constrained geometry of $\mathbb{Q}_p$.

- In Euclidean space $\mathbb{R}^n$, the maximum number of mutually equidistant points is $n+1$, e.g., an equilateral triangle in $\mathbb{R}^2$, or a regular tetrahedron in $\mathbb{R}^3$.
- In contrast, in $\mathbb{Q}_p$, the structure is more rigid. Due to the ultrametric property, distances do not "add" linearly, and only a finite number of values are allowed for leading digits.
- Because there are only p possible initial digits modulo p, the pigeonhole principle bounds the number of equidistant points to at most p.
- For example:
  - If $p = 2$, at most 2 points can be equidistant.
  - If $p = 3$, at most 3 such points can exist.
  - Larger p allows more equidistant configurations.

Conclusion: Geometry over $\mathbb{Q}_p$ is highly structured and limited. This "tightness" makes p-adic spaces fundamentally different from Euclidean spaces in their geometric behavior.

CONCLUSION

An Overview of the p-adic Numbers

A fresh and comprehensive viewpoint on number systems and analysis is provided by the theory of p-adic numbers. p-adic numbers originate from a valuation based on divisibility by a prime number p, in contrast to the real number system, which is based on the typical absolute value. A fundamentally different metric and topology result from this shift in perspective, and this leads to unexpected and beautiful mathematical properties.

As this introduction has shown, the rationals $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$ are completed to construct $\mathbb{Q}_p$, the field of p-adic numbers. Similar to how $\mathbb{R}$ is the completion of $\mathbb{Q}$ under the standard Euclidean metric, $\mathbb{Q}_p$ is a complete, completely unrelated, non-Archimedean field. We looked at important ideas like the p-adic absolute value, which creates an ultrametric space, and the p-adic valuation $v_p(x)$, which calculates the power of p dividing a rational number x. The geometry of $\mathbb{Q}_p$ is drastically changed by the ultrametric inequality (strong triangle inequality), which states that all triangles in this space are isosceles or equilateral and all balls are both open and closed (clopen).

Every element in $\mathbb{Q}_p$ has a distinct p-adic expansion, which is comparable to decimal expansions in the real numbers but extends infinitely to the right rather than the left. It was demonstrated that the structure of $\mathbb{Z}_p$, the ring of p-adic integers, is compact. These extensions make calculations easier and shed light on number-theoretic issues. Furthermore, a thorough investigation of the geometry of $\mathbb{Q}_p$ revealed characteristics like the lack of dense equidistant point configurations outside of p elements and unique behavior of distances and point clusters. Applications of this geometry can be found in algebraic geometry, number theory, and contemporary arithmetic.

In the end, the p-adic numbers contribute to mathematics by providing a different, but no less rigorous, framework for arithmetic and analysis. They are essential to contemporary number theory, particularly in the areas of local-global principles, Diophantine equations, and the demonstration of important theorems like Hensel's lemma, the local-global principle, and portions of Fermat's Last Theorem. Mathematicians can access tools and insights not visible in real or complex analysis by cultivating an intuition for the p-adic world. Thus, the p-adic numbers provide a strong lens through which to view classical problems from a different—and frequently more illuminating—angle.

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