Mahapurusha Srimanta Sankaradeva Viswavidyalaya

Department of Mathematics

# DISSERTATION

on

# “BASIC RESULTS OF HYBRID DIFFERENTIAL EQUATIONS”

**Submitted in partial fulfillment of the requirements for M.Sc in Mathematics Masters of science**


**Submitted by:**

Aminur Islam

Roll No.: MAT-18/23

Registration No.: MSSV-0023-101-001341

Session: 2023-2025

**Under the supervision of:**

Dr. Raju Bordoloi

HOD, Department of Mathematics

MSSV, Nagaon

**Mahapurusha Srimanta Sankaradeva Viswavidyalaya**

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## Certificate

This is to certify that the dissertation entitled “Basic result of hybrid differential equations” submitted by Aminur Islam, Roll No. MAT-18/23, Registration No. MSSV-0023-101-001341 , in partial fulfilment for the award of the degree of Master of Science in Mathematics, is a bonafide record of original work carried out under my supervision and guidance.

To the best of my knowledge, the work has not been submitted earlier to any other institution for the award of any degree or diploma.

**Dr. Raju Bordoloi**

Associate Professor

Department of Mathematics

Mahapurusha Srimanta Sankaradeva Viswavidyalaya

Date:

Place: Signature of Guide

# Declaration

I, Aminur Islam, hereby declare that the dissertation titled “Basic result of hybrid differential equations”, submitted to the Department of mathematics , Mahapurusha Srimanta Sankaradeva Viswavidyalaya, is a record of original work carried out by me under the supervision of Dr. Raju Bardoloi, Associate professor.

This work has not been submitted earlier to any other institution or university for the award of degree or diploma.

Place: Aminur Islam

Date: Roll No.: MAT-18/23

# Acknowledgement

First and foremost, I would like to express my sincere gratitude to my gude, Dr.Raji Bordoloi, Associate Professor, Department of Mathematics,Mahapurusha Srimanta Sankaradeva Viswavidyalaya, for his valuable guidance, continuous support, and encouragement throughout the course of this dissertation.

I also extend my heartleft thanks to the faculty members of the Department of Mathematics for their constant academic support and the friendly learning environment they provided.

I am deeply grateful to my family and friends for their unwavering moral support and motivation throughout my academic journey. Their encouragement gave me the strength to successfully complete this work.

Lastly, I thank all those who directly or indirectly helped me during this project.

Place: Aminur Islam

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# Contents

# Chapter 1

# Preliminaries

## 1.1 Introduction

This paper is concerned with an existence theorem for hybrid nonlinear differential equations under mixed Lipschitz and Carathéodory conditions. In recent years, quadratic perturbations of nonlinear differential equations have attracted much attention. We call such differential equations hybrid differential equations. The existence theory for such hybrid equations can be developed using hybrid fixed point theory.

## 1.2 Nonlinear Differential Equation

A linear differential equation is a differential equation that is defined by a linear polynomial in the unknown function and its derivatives, that is an equation of the form

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \ldots + a_n(x)y^{(n)} = b(x),$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ are arbitrary differentiable functions that do not need to be linear, and $y', y'', \ldots, y^{(n)}$ are the successive derivatives of an unknown function $y$ of the variable $x$. Otherwise, they are called **nonlinear** differential equation.

## 1.3 Quadratic Perturbation

Perturbation techniques or methods are very much useful in the subject of nonlinear analysis for studying the dynamical systems represented by nonlinear differential and integral equations in a nice way. Sometimes a differential equation representing a certain dynamical system is not easily solvable or analyzed, however, the perturbation of such problem in someone manner makes it possible to study the problem with available methods for different aspects of the solutions. To be more specific, for any closed andboundedinterval $J = [0, T]$ of the real line $\mathbb{R}$, consider the initial value problem of nonlinear first order ordinary differential equations,

$$\left.\begin{aligned} x'(t) &= f(t, x(t)) \ \text{ a.e. } \ t \in J \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$.

If we split the function $f$ into sum of two functions $f_1$ and $f_2$, that is, $f = f_1 + f_2$, then these functions have some nice properties and the nonlinear differential equation

$$\left.\begin{aligned} x'(t) &= f_1(t, x(t)) + f_2(t, x(t)) \ \text{ a.e. } t \in J \\ x(0) &= x_0 \in \mathbb{R} \end{aligned}\right\} \tag{1.2}$$

is easily solvable with the available functional theoretic techniques. The method of doing so is called perturbation method and the differential equation (1.2) is called a perturbation of the differential equation

$$\left.\begin{aligned} x'(t) &= f_1(t, x(t)) \ \text{ a.e. } t \in J \\ x(0) &= x_0 \in \mathbb{R} \end{aligned}\right\} \tag{1.3}$$

The above differential equation (1.2) is obtained by perturbing the nonlinearity $f_1$ from (1.3) and is called a perturbed differential equation.

A perturbation which involves the multiplication or division by a term is called a quadratic perturbation. The differential equation

$$\left.\begin{aligned} \tfrac{d}{dt}\left[\tfrac{x(t)}{f_2(t,x(t))}\right] &= f_1(t, x(t)) \ \text{ a.e. } t \in J \\ x(0) &= x_0 \in \mathbb{R} \end{aligned}\right\} \tag{1.4}$$

is a quadratic perturbation for the differential equations (1.3)

## 1.4 Lipschitz Condition

Let $f$ be such that $\frac{\partial f}{\partial y}$ exists and is bounded for all $(x, y) \in D$, where $D$ is a domain or closed domain such that the line segment joining any two points of $D$ lies entirely within $D$, that is, for all $(x, y_1), (x, y_2) \in D$, there exists $k > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leqslant k|y_1 - y_2|,$$

Then $f$ is said to satisfy a **Lipschitz condition** (with respect to $y$) in $D$, where the **Lipschitz constant** is given by

$$k = \sup_{(x,y)\in D} \left|\frac{\partial f(x, y)}{\partial y}\right|$$

.

## 1.5 Carathéodory

A function $g : E(\subseteq \mathbb{R}^2) \to \mathbb{R}$ is said to satisfy the **Carathéodory condition** if

(a) $g(t, u)$ is continuous in $u$ for each fixed $t$ and **Lebesgue measurable** in $t$ for each fixed $u$.

(b) $M(t)$ is a summable function on $[t_0, t_0 + 1]$ and

$$|g(t, u)| \leqslant M(t), \quad (t, u) \in E.$$

## 1.6 Banach Algebra

A non-empty set $\mathcal{A}$ is called an **Algebra** if

(a) $(\mathcal{A}, +, \cdot)$ is a vector space over a field $\mathbb{F}$.

(b) $(\mathcal{A}, +, \circ)$ is a ring and

(c) $(\alpha a) \circ b = \alpha(a \circ b) = a \circ (\alpha b)$ for every $\alpha \in \mathbb{F}$, for every $a, b \in \mathcal{A}$.

$\mathcal{A}$ is a **commutative algebra** if $(\mathcal{A}, +, \circ)$ is commutative.

If $\mathcal{A}$ is an algebra and $\|\cdot\|$ is a norm on $\mathcal{A}$ satisfying

$$\|ab\| \leqslant \|a\|\,\|b\|, \quad \text{for all} \;\; a, b \in \mathcal{A},$$

then $(\mathcal{A}, \|\cdot\|)$ is called a **Normed Algebra**.

A complete normed algebra is called a **Banach Algebra**.

# Chapter 2

# Hybrid differential equation

The quadratic perturbations of nonlinear differential equations are called **hybrid differential equations**.

Let $\mathbb{R}$ be the real line and $J = [t_0, t_0 + a)$ be bounded interval in $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$. Let $C(J \times \mathbb{R}, \mathbb{R})$ denote the class of continuous functions $f : J \times \mathbb{R} \to \mathbb{R}$ and let $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ denote the class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ such that

(a) the map $t \mapsto g(t, x)$ is measurable for each $x \in \mathbb{R}$, and

(b) the map $x \mapsto g(t, x)$ is continuous for each $t \in J$.

The class $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on $J$.

The DE, we are interested in, is the following Hybrid Differential equation (HDE)

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] &= g(t, x(t)) \;\; a.e.\; t \in J \\ x(t_0) &= x_0 \in \mathbb{R} \end{aligned}\right\} \tag{2.1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$.

By solution of the HDE $(**)$, we mean a function $x \in AC(J, \mathbb{R})$ such that

(a) the function $t \mapsto \frac{x}{f(t,x)}$ is absolutely continuous for each $x \in \mathbb{R}$.

(b) $x$ satisfies the equations in (2.1).

where $AC(J,\mathbb{R})$ is the space of absolutely continuous real-valued functions defined on $J$.

# Chapter 3

# Existence result

In this chapter, we prove the existence results for the HDE (2.1) on the closed and bounded interval $J = [t_0, t_0+a)$ under mixed Lipschitz and Carathéodory conditions on the nonlinearities involved in it. We place the HDE (2.1) in the space $C(J, \mathbb{R})$ of continuous real-valued functions defined on $J$. We define a supremum norm $||\cdot||$ in $C(J, \mathbb{R})$ defined by

$$||x|| = \sup_{t \in J} |x(t)|$$

and a multiplication in $C(J, \mathbb{R})$ by

$$(xy)(t) = x(t)y(t)$$

for $x, y \in C(J, \mathbb{R})$. Then $C(J, \mathbb{R})$ is a Banach algebra with respect to above norm and multiplication in it. $L^1(J, \mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on $J$ equipped with the norm $||\cdot||_{L^1}$ defined by

$$||x||_{L^1} = \int_{t_0}^{t_0+a} |x(s)| ds.$$

**Theorem 3.0.1.** *[5] Let $S$ be a non-empty, closed convex and bounded subset of a Banach algebra $X$ and let $A : X \to X$ and $B : S \to X$ be two operators such that*

*(a) $A$ is $\mathcal{D}$-*`Lipschitz` *with $\mathcal{D}$-function $\psi$,*

*(b) $B$ is completely continuous,*

*(c) $x = AxBy \implies x \in S$ for all $y \in S$, and*

*(d) $M\psi(r) < r$, where $M = ||B(S)|| = \sup\{||Bx|| : x \in S\}$.*

*Then the opertor equation $AxBx = x$ has a solution in $S$.*

We consider the following hypotheses in what follows.

$(A_0)$ The function $x \mapsto \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.

$(A_1)$ There exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leqslant L|x-y|$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

$(A_2)$ There exists a function $h \in L^1(J, \mathbb{R})$ such that

$$|g(t,x)| \leqslant h(t) \quad \text{a.e.} \quad t \in J$$

for all $x \in \mathbb{R}$.

**Lemma 3.0.2.** *Suppose that the map $x \mapsto \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ a.e. for $t \in J$. Then for any $h \in L^1(J, \mathbb{R}_+)$, the function $x \in AC(J, \mathbb{R}_+)$ is a solution of the HDE*

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = h(t) \quad a.e.\ t \in J \tag{3.1}$$

*and*

$$x(0) = x_0 \tag{3.2}$$

*if and only if $x$ satisfies the hybrid integral equation (HIE)*

$$x(t) = [f(t, f(t))]\left(\frac{x_0}{f(t_0, x_0)} + \int_{t_0}^{t} h(s)ds\right), \quad t \in J. \tag{3.3}$$

**Proof:** Let $h \in L^1(J, \mathbb{R}_+)$. Assume that $x$ is a solution of the HDE (3.1)-(3.2). By definition $\frac{x(t)}{f(t,x(t))}$ is absolutely continuous, and so, almost everywhere differentiable, whence $\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right]$ is Lebesgue integrable on $J$. Applying integration to (3.1) from $t_0$ to $t$, we obtain the HDE (3.3) on $J$.

Conversely, assume that $x$ satisfies the QHIE (3.3). Then by direct differentiation, we obtain (3.1). Again, substituting $t = t_0$ in (3.3) yields

$$\frac{x(t_0)}{f(t_0, x(t_0))} = \frac{x_0}{f(t_0, x_0)}.$$

Since the mapping $x \mapsto \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$, the mapping $x \mapsto \frac{x_0}{f(t_0,x)}$ is injective in $\mathbb{R}$, whnce $x(t_0) = x_0$. □

**Theorem 3.0.3.** *Assume that*

*(i) The map* $x \mapsto \frac{x}{f(t,x)}$ *is increasing in* $\mathbb{R}$ *a.e. for* $t \in J$.

*(ii) There is a constant* $L > 0$ *such that*

$$|f(t,x) - f(t,y)| \leqslant L|x - y|$$

*for all* $t \in J$ *and* $x, y \in \mathbb{R}$.

*(iii) There is a function* $h \in L^1(J, \mathbb{R})$ *such that*

$$|g(t,x)| \leqslant h(t) \quad a.e.\ t \in J$$

*for all* $x \in \mathbb{R}$.

*Further, if*

$$L\left(\left|\frac{x_0}{f(t_0,x_0)}\right| + \|h\|_{L^1}\right) < 1 \tag{3.4}$$

*then the HDE (2.1) has a solution on* $J$.

**Proof:** We take $X = C(J, \mathbb{R})$ and define a subset $S$ of $X$ defined by

$$S = \{x \in X : \|x\| \leqslant N\} \tag{3.5}$$

where $N = \frac{F_0\left(\left|\frac{x_0}{f(t_0,x_0)}\right| + \|h\|_{L^1}\right)}{1 - L\left(\left|\frac{x_0}{f(t_0,x_0)}\right| + \|h\|_{L^1}\right)}$ and $F_0 = \sup_{t \in J} |f(t,0)|$.

Then $S$ is closed, convex and bounded subset of the Banach space $X$. Now, using hypotheses (i), (iii) and applying lemma (3.0.2), the HDE (2.1) is equivalent to the HIE

$$x(t) = [f(t, x(t))]\left(\frac{x_0}{f(t_0,x_0)} + \int_{t_0}^{t} g(s, x(s))ds\right) \tag{3.6}$$

for $t \in J$.

We define two operators $A : X \to X$ and $B : S \to X$ by

$$Ax(t) = f(t, x(t)), \quad t \in J \tag{3.7}$$

and

$$Bx(t) = \frac{x_0}{f(t_0,x_0)} + \int_{t_0}^{t} g(s, x(s))ds \tag{3.8}$$

The the HIE (3.6) is transformed into the opertor equation

$$Ax(t)Bx(t) = x(t), \quad t \in J \tag{3.9}$$

We now claim the following

(a) $A$ is a Lipschitz operator on $X$ with Lipschitz constant $L$.

(b) $B$ is completely continuous on $S$.

(c) $x = AxBy \implies x \in S \;\; \forall y \in S$.

(d) hhh

To show (a): Let $x, y \in X$. Then by hypothesis (ii),

$$|Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \leqslant L|x(t) - y(t)| \leqslant \|x - y\| \; \forall t \in J$$

Taking supremum over $t$, we obtain

$$\|Ax - Ay\| \leqslant L\|x - y\| \quad \forall x, y \in X$$

To show (b): We first show that $B$ is continuous on $S$. Let $\{x_n\}$ be a sequence in $S$ such that $x_n \to x$ as $n \to \infty$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty}\left(\frac{x_0}{f(t_0, x_0)} + \int_{t_0}^{t} g(s, x_n(s))ds\right) \\
&= \frac{x_0}{f(t_0, x_0)} + \lim_{n\to\infty}\int_{t_0}^{t} g(s, x_n(s))ds \\
&= \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^{t}\left(\lim_{n\to\infty} g\left(s, x_n(s)\right)\right)ds \\
&= \frac{x_0}{f(t_0, x_0)} + \int_{t_0}^{t} g(s, x(s))ds \\
&= Bx(t) \quad \forall t \in J
\end{aligned}$$

Thus $B$ is a continuous operator on $S$.

Next, we show that $B$ is a compact operator on $S$. For this, it is sufficient to show that $(b_1)$

$B(S)$ is uniformly bounded and equi-continuous in $X$. Let $x \in S$. Then by hypothesis (iii),

$$\begin{aligned} |Bx(t)| &\leqslant \left|\frac{x_0}{f(t_0, x_0)}\right| + \int_{t_0}^{t} |g(s, x(s))| ds \\ &\leqslant \left|\frac{x_0}{f(t_0, x_0)}\right| + \int_{t_0}^{t} h(s) ds \\ &\leqslant \left|\frac{x_0}{f(t_0, x_0)}\right| + \|h\|_{L^1} \quad \forall t \in J \end{aligned}$$

Taking supremum over $t$,

$$\|Bx\| \leqslant \left|\frac{x_0}{f(t_0, x_0)}\right| + \|h\|_{L^1} \quad \forall x \in S$$

Thus $B(S)$ is uniformly bounded. Now we show that $B(S)$ is equi-continuous in $X$. Let $t_1, t_2 \in J$. Then for any $x \in S$, we have

$$\begin{aligned} |Bx(t_1) - Bx(t_2)| &= \left|\int_{t_0}^{t_1} g(s, x(s)) ds - \int_{t_0}^{t_2} g(s, x(s)) ds\right| \\ &\leqslant \left|\int_{t_2}^{t_1} |g(s, x(s))| ds\right| \\ &\leqslant |p(t_1) - p(t_2)| \quad \text{where } p(t) = \int_{t_0}^{t} h(s) ds \end{aligned}$$

Since $p$ is a continuous function on the compact set $J$, it is uniformly continuous too. So given $\varepsilon > 0, \exists\, \delta > 0$ such that

$$|t_1 - t_2| < \delta \implies |Bx(t_1) - Bx(t_2)| < \varepsilon \quad \forall t_1, t_2 \in J \text{ and } \forall x \in S$$

i.e. $B(S)$ is equi-continuous in $X$. Thus, by Arzelá-Ascoli theorem, $B(S)$ is compact. Hence $B$ is completely continuous on $S$.

To show (c): Let $x \in X$ and $y \in S$ be such that $x = AxBy$. Then

$$\begin{aligned} |x(t)| &= |Ax(t)||By(t)| \\ &= |f(t, x(t))| \left|\frac{x_0}{f(t_0, x_0)} + \int_{t_0}^{t} g(s, y(s)) ds\right| \\ &\leqslant (|f(t, x(t)) - f(t, 0)| + |f(t, 0)|) \left(\left|\frac{x_0}{f(t_0, x_0)}\right| + \int_{t_0}^{t} |g(s, y(s))| ds\right) \\ &\leqslant (L|x(t)| + F_0) \left(\left|\frac{x_0}{f(t_0, x_0)}\right| + \int_{t_0}^{t} |h(s)| ds\right). \text{ where } F_0 = \sup_{t \in J} |f(t, 0)| \\ &\leqslant \frac{F_0 \left(\left|\frac{x_0}{f(t_0, x_0)}\right| + \|h\|_{L^1}\right)}{1 - L\left(\left|\frac{x_0}{f(t_0, x_0)}\right| + \|h\|_{L^1}\right)} \end{aligned}$$

Taking supremum over t,

$$\|x\| \leqslant \frac{F_0\left(\left|\frac{x_0}{f(t_0,x_0)}\right| + \|h\|_{L^1}\right)}{1 - L\left(\left|\frac{x_0}{f(t_0,x_0)}\right| + \|h\|_{L^1}\right)} = N$$

To show (d): We have

$$M = \|B(S)\| = \sup\{\|Bx\| : x \in S\} \leqslant \left|\frac{x_0}{f(t_0, x_0)}\right| + \|h\|_{L^1}$$

and so,

$$\alpha M \leqslant L\left(\left|\frac{x_0}{f(t_0, x_0)}\right| + \|h\|_{L^1}\right) < 1$$

Thus, all the conditions of Theorem 3.0.1 are satisfied and hence the operator equation $AxBx = x$ has a solution in $S$. As a result, the HDE (2.1) has a solution defined on $J$. □

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