A COMPREHENSIVE STUDY OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

Dissertation submitted to the Mahapurusha Srimanta Sankaradeva Viswavidyalaya, Nagaon for the partial fulfillment of the degree of Master of Science in Mathematics.

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CERTIFICATE

This is to certify that RUHANI SULTANA BORBHUYAN bearing Roll No MAT-16/23 and Redg. No. MSSV-0023-101-001350 has prepared her dissertation entitled "A COMPREHENSIVE STUDY OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS" submitted to the Department of Mathematics, MAHAPURUSHA SRIMANTA SANKARADEVA VISWAVIDYALAYA, Nagaon, for fulfillment of M.Sc. degree, under guidance of me and neither the dissertation nor any part thereof has submitted to this or any other university for a research degree or diploma.

He/She fulfilled all the requirements prescribed by the department of Mathematics.

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DECLARTION

I hereby declare that this dissertation entitled "**A Comprehensive Study of First-Order Ordinary Differential Equations**" submitted to **MAHAPURUSHA SRIMANTA SANKARADEVA VISWAVIDYALAYA** for the award of the degree of Master of Science in Mathematics, is my original work and has not been submitted previously by me or any other person for any other degree or diploma at this or any other institution.

This work has been carried out under the guidance of **DR. MIRA DAS**, Assistant professor, Department of Mathematics, **MAHAPURUSHA SRIMANTA SANKARADEVA VISWAVIDYALAYA.** I further declare that all sources of information and data used in the preparation of this dissertation have been duly acknowledged.

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Yours sincerely

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ABSTRACT

This dissertation provides an in-depth exploration of *First-Order Ordinary Differential Equations (ODEs),* which are mathematical equations involving a function and its first derivative. These equations form the foundation of mathematical modeling in various fields such as physical, engineering, biology, economics, and environmental sciences. The study begins with a formal definition and classification of first-order ODEs into linear and nonlinear types, autonomous and non-autonomous equations, as well as separable, exact, and homogeneous forms.

A central focus of the dissertation is on **analytical solution methods.** Detailed procedures for solving different categories of first-order ODEs are presented, including the method of separation of variables, the integrating factor method, and techniques for solving exact and non-exact equations. The conditions for existence and uniqueness of solutions, based on the Picard-Lindelof theorem, are also discussed to provide a theoretical underpinning for the solution methods.

The study further investigates the **qualitative behavior** of solutions through graphical and phase line analysis, helping to understand the long-term behavior of systems modeled by first-order ODEs. In addition, **numerical methods** such as Euler's method are introduced for cases where analytical solutions are difficult or impossible to obtain. These methods are illustrated through step-by-step examples to demonstrate their practical utility.

Real-life **applications** of first-order ODEs are emphasized throughout the dissertation to showcase their relevance in solving practical problems. Case studies include modeling exponential population growth and decay, Newton's law of cooling, electrical circuits involving resistors and capacitors (RC circuits), and chemical reaction rates. Each application is accompanied by a mathematical model, solution, and interpretation of the results.

Moreover, the dissertation addresses **challenges and limitations** associated with first-order ODEs, such as sensitivity to initial conditions and the complexity of certain nonlinear equations. Finally, the study highlights the importance of first-order ODEs as a stepping stone to more advanced differential equations and systems, emphasizing their role in both theoretical research and applied sciences.

Through a combination of theory, solution methods, graphical representation, numerical approximation, and real-world application, this dissertation aim to offer a holistic understanding of first-order ordinary differential equations. It serves both as a rigorous academic inquiry and a practical guide for applying ODEs in various scientific contexts.

LITERATURE REVIEW

FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

Ordinary Differential Equations (ODEs), particularly first-order ODEs, have been fundamental to mathematical modeling across physical, biological, and engineering sciences. A first-order ODE involves a function and its first derivative, offering a foundational tool for analyzing dynamic systems that change over time.

The study of first-order ODEs dates back to the 17th century, with pioneering contributions from mathematicians such as Isaac Newton and Gottfried Wilhelm Leibniz. Their early work laid the foundation for formulating and solving differential equations in analytical form. The linear first-order ODEs, expressed in the form $\frac{dy}{dx} + P(x)y = Q(x)$, were among the first to be systematically solved using the integrating factor method. This method remains a cornerstone in solving linear equations.

The numerical analysis of Odes, particularly after the advent of digital computing, opened up new possibilities. Euler's method, Runge-kutta methods, and other numerical schemes become essential for approximating solutions to first-order ODEs where analytical methods fail. These have been rigorously analyzed for stability, convergence, and accuracy.

Zill (2018) provides a comprehensive introduction to differential equations, with a stong emphasis on boundary-value problems. Tenenbaum and Pollard (1985) present a classical approach in *Ordinary Differential Equations,* focusing on analytical methods, including separation of variables and integrating factors, which are essential tools for solving first-order ODEs. Arnold (1992), in his work *Ordinary Differential Equations,* explores the geometric interpretation and qualitative analysis of ODEs. Polking (1995) introduces the use of MATLAB in solving ODEs numerically. Logan (2015) emphasized a balanced approach between theory and applications in *A First Course in Differential Equations.*

Recent literature continues to focus on both analytical and numerical approaches, with increasing interest in applications. For instance, first-order ODEs are widely used in modeling population growth (Malthusian and logistic models), radioactive decay, Newton's law of cooling, and in circuits governed by Kirchhoff's laws.

Modern research also investigates fractional differential equations and their relation to memory-dependent systems, extending the concept of ordinary derivatives to fractional orders. Moreover, advances in software such as MATLAB, Mathematica, and Python libraries like SciPy have revolutionized the way differential equations are analyzed and visualized.

In summary, the literature on first-order ordinary differential equations demonstrates a rich history of analytical techniques, a broadening of qualitative and numerical analysis, and a growing emphasis on interdisciplinary applications. This body of work provides a solid foundation and ongoing inspiration for the continued development of methods and models based on first-order ODEs.

CONTENT

INTRODUCTION

Ordinary Differential Equations (ODEs) form the cornerstone of mathematical modeling in the natural and applied science. Among these first-order ODEs- differential equations involving only the first derivative of the unknown function-occupy a central place due to their simplicity and broad applicability. These equations describe a wide range of phenomena where the rate of change of a quality is proportional to the quantity itself or some function of it.

This dissertation aims to provide a comprehensive examination of first-order ODEs, beginning with theoretical fundamentals and progressing to various methods of solution, both analytical and numerical. We will also explore qualitative techniques for analyzing the behavior of solutions when closed-form expressions are difficult or impossible to obtain.

Emphasis will be placed on applications across disciplines, demonstrating how even simple differential equations can yield deep insights in the behavior of physical, biological, and economic systems. Through this, we hope to underscore the versatility and utility of first-order ODEs in understanding and predicting the dynamics of real-world systems.

The origin of Ordinary Differential Equations (ODEs) in deeply rooted in the history of mathematics and physics. The development of ODEs can be traced back to the 17^{th} century, when scientists and mathematicians sought to describe natural phenomena using mathematical equations.

Key Milestones in the origin of ODEs:-

1) Ancient and Medieval Contributions:-
    - Early forms of differential reasoning appeared in Greek mathematics, particularly in the works of Archimedes (Circa 287-212 BCE) and Apollonius of Perga (Circa 262-190 BCE).

- Indian and Islamic mathematicians also made contributions to early calculus-like ideas.

2) Renaissance and the Birth of calculus (17^{th} century):-
   - The formal development of ODEs began with Rene Descartes (1596-1650) and Pierre de fermat (1601-1665), who introduced early ideas of analytic geometry.
   - Isaac Newton (1643-1727) and Gottfried Wilhelm Leibniz (1646-1716) independently developed calculus, providing the foundation for differential equations.
   - Newton introduced differential equations in his works on motion, particularly in Principia Mathematica (1687).
3) 18^{th} century – Systematic Development:-
   - Jacob Bernoulli (1655-1705) and Johann Bernoulli (1667-1748) contributed to solving specific ODEs.
   - Leonhard Euler (1707-1783) made significant advancements in solving first-order and second-order differential equations and introduced the Euler method.
   - Jean Le Rond d'Alembert (1717-1783) and Joseph-Louis Lagrange (1736-1813) further developed techniques in mechanics and variational calculus.
4) 19^{th} and 20^{th} century – Modern Theory:-
   - Carl Gustav Jacobi (1804-1851) and Simeon Denis Poisson (1781-1840) worked on applications in mechanics.
   - Sophus Lie (1842-1899) introduced group theory to study differential equations systematically.
   - The 20^{th} century saw the rise of qualitative theory, Pioneered by Henri Poincare (1854-1912), focusing on stability and chaos in ODEs.

The origin of ordinary differential equations is closely linked with the evolution of calculus and mathematical physics. Today, ODEs are fundamental in modeling real-world phenomena in engineering, economics, and biological science.

CHAPTER-1

PRELIMINARIES

i. INTRODUCTIONS:

In this chapter we will discuss about the basic concepts First-Order Ordinary Differential Equations as well as the Order and Degree of a First-Order Differential Equation etc.

ii. DEFINITION AND BASIC CONCEPTS OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATION:

Ordinary Differential Equations (ODEs) are fundamental tools in mathematics used to describe the relationship between a function and its derivatives. A first-order ordinary differential equation involving a function $y(x)$ and its first derivative $\frac{dy}{dx}$, but no higher derivatives.

The general form of a first-order ODE is:

$$\frac{dy}{dx} = f(x, y)$$

This equation expresses the rate of change of a dependent variable y with respect to an independent variable x, as a function of both x and y. Solving such an equation involves finding a function $y(x)$ that satisfies the given relationship.

BASIC CONCEPTS:-

i. Order: The order of a differential equation is the highest derivative present. For first-order ODEs, only the first derivative $\frac{dy}{dx}$ appears.
ii. Solution: A solution is a function $y = \emptyset(x)$ that satisfies the ODE in some interval.
iii. General Solution: The general solution contains an arbitrary constant and represents a family of solutions.
iv. Particular Solution: A solution obtained by assigning specific values to the arbitrary constants often found using initial conditions.

v. Initial Value Problem (IVP): A problem that includes a differential equation and an initial condition, typically:

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

COMMON TYPES OF FIRST-ORDER ODEs

i. Separable Equations:

$$\frac{dy}{dx} = g(x)h(y)$$

$$\frac{1}{h(y)}dy = g(x)dx$$

ii. Linear Equations:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\mu(x) = e^{\int p(x)dx}$$

iii. Exact Equations:

$$M(x, y)dx + N(x, y) = 0$$

$$\frac{dM}{dy} = \frac{dN}{dx}$$

iv. Homogeneous Equations: Function $f(x, y)$ is homogeneous of degree 0: $\frac{dy}{dx} = f(\frac{y}{x})$

1.3 ORDER AND DEGREE OF A FIRST-ORDER ORDINARY DIFFERENTIAL EQUATION:

A differential equation is an equation involving a function and its derivatives. The order and degree are two fundamental characteristics used to classify differential equations.

I. Order: The order of a differential equation is the highest derivative of the unknown function present in the equation.

For first-order ODEs, the highest derivative is the first derivate $\frac{dy}{dx}$, so that is 1.

II. Degree: The degree of a differential equation is the exponent of the highest-order derivative, after the equation is free from radicals and fractions involving derivatives.

For a first-order ODE the degree is the power of $\frac{dy}{dx}$ after making the equation polynomial in derivatives.

Equation 1:

$$\left(\frac{dy}{dx}\right)^2 + y = x$$

Degree = 2 (power of the first derivative)

Equation 2:

$$\frac{dy}{dx} + y = e^x$$

Degree = 1 (first derivative appears to the power of 1)

Equation 3:

$\sqrt{\frac{dy}{dx}} + y = x$, not defined degree until it's written as a polynomial in derivatives. (In this case, degree is not defined in the usual sense because the derivative is inside a root.)

CHAPTER-2

ANALYTICAL SOLUTION METHODS

2.1 INTRODUCTION: Analytical method provide exact solutions for first-order ODEs using techniques like separation of variables, integrating factors, and substitutions.

2.2 SEPARABLE EQUATIONS:

A first-order differential equation is separable if it can be written in one of the following forms

$$\frac{dy}{dx} = f(x,y) = \frac{g(x)}{h(y)}$$

$$\frac{dy}{dx} = f(x,y) = \frac{h(y)}{g(x)}$$

SOLVING SEPARABLE EQUATION:

A separable equation is solved by separating the variable, that is, rearranging the equation so that everything involving y appears on one side of the equation, and everything involving x appears on the other. The equation can then be integrated directly.

For an equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

Multiplying both sides by $h(x)dx$ gives:

$$h(y)dy = g(x)dx$$

Which can be integrated directly:

$$\int h(y)dy = \int g(x)dx$$

This will yield a solution for $y(x)$

Similarly, an equation in the form:

$$\frac{dy}{dx} = \frac{h(y)}{g(x)}$$

Can be multiplied by $\frac{dx}{h(y)}$ and then integrated:

$$\int \frac{dy}{h(y)} = \int \frac{dx}{g(x)}$$

Thus yielding a solution for $y(x)$.

NOTE: The solution obtained for y by computing these integrals maybe implicitly defined. Rearranging the solution may be necessary to obtain an explicit solution for y although in some cases it may not be possible to express explicitly.

EXAMPLE: Solve the differential equation

$$\frac{dy}{dx} = \frac{1}{x+3}$$

Subject to the condition $y(0) = 1$

Solution: Multiplying both sides of the equation by dx gives $\frac{dy}{y} = \frac{dx}{x+3}$

Every term involving x now appears on the right hand side, and every term involving y now appears on the left-hand side. Each side of the equation can therefore be integrated directly:

$$\int \frac{dy}{y} = \int \frac{dx}{x+3}$$

$$\ln|y| = \ln|x+3| + c$$

To find a solution which gives y explicitly, exponentiate both sides to obtain :

$$y = A(x+3)$$

Where A is an arbitrary constant.

NOTE: The constant A comes from applying the laws of logarithms and powers:

$$e^{\ln|x+3|+c} = e^c e^{\ln|x+3|} = e^c(x+3) = A(x+3)$$

Where, $A = e^c$, Since e is a number, e raised to a constant power is also a constant, and it is permissible to denote that constant by a single letter A.

Hence, the general solution to the differential equation is $y = A(x+3)$

To find the solution to the satisfies $y(0) = 1$, substitute $x = 0$ and $y = 1$ into the solution and solve for A:

$$1 = A(0+3)$$

$$\Rightarrow 3A = 1$$

$$\Rightarrow A = \frac{1}{3}$$

Hence, the solution to the given differential equation satisfies the condition $y(0) = 1$ is

$$y = \frac{1}{3}(x+3)$$

2.3 **LINEAR FIRST-ORDER EQUATIONS**:

A first-order linear differential equation is one that can be put into the form

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \cdots\cdots\ \cdots\cdots\cdots \to 1$$

Where P and Q are continuous functions on a given interval. This type of equation occurs frequently in various sciences, as we will see.

An example of a linear equation is $xy' + y = 2x$ because, for $x \neq 0$, it can be written in the form

$y' + \frac{1}{x}y = 2$ ……………… → 2

Notice that this differential equation is not separable because it's impossible to factor the expression for y' as a function of x times a function of y. But we can still solve the equation by noticing, by the Product Rule, that

$$xy' + y = (xy')$$

And so we can rewrite the equation as

$$(xy)' = 2x$$

If we now integrate both sides of this equation, we get

$xy = x^2 + C$ or $y = x + \frac{C}{x}$

If we had been given the differential equation in the form of Equation 2, we would have had to take the preliminary step of multiplying each side of the equation by x.

It turns out that every first-order linear differential equation can be solved in a similar fashion by multiplying both sides of Equation 1 by a suitable function $I(x)$ called an integrating factor. We try to find I so that the left side of Equation 1, when multiplied by $I(x)$, becomes the derivative of the product I(x)y:

$$I(x)\{y' + P(x)\} = \{I(x)y\}' \qquad \text{……………} \rightarrow 3$$

If we can find such a function I then Equation 1 becomes

$$\{I(x)y\}' = I(x)Q(x)$$

Integrating both sides, we would have

$$\mathrm{I(x)y} = \int \mathrm{I(x)Q(x0dx + C}$$

So the solution would be

$$y(x) = \frac{1}{I(x)}\left[\int I(x)Q(x)dx + C\right] \qquad \text{……………} \rightarrow 4$$

To find such an I, we expand Equation 3 and cancel terms:

$$I(x)y' + I(x)P(x)y = \{I(x)y\}' = I'(x)y + I(x)y'$$

$$I(x)P(x) = I'(x)$$

This is a separable differential equation for I, which we solve as follows:

$$\int \frac{dI}{I} = \int P(x)dx$$

$$\ln|I| = \int P(x)dx$$

$$I = Ae^{\int P(x)dx}$$

Where $A = \pm e^c$. We are looking for a particular integrating factor, not the most general one, so we take $A = 1$ and use

$$I(x) = e^{\int P(x)dx} \qquad \cdots\cdots\cdots\cdots\cdots \rightarrow 5$$

Thus a formula for the general solution to Equation 1 is provided by Equation 4, where I is given by Equation 5. Instead of memorizing this formula, however, we just remember the form of the integrating factor.

EXAMPLE:

Solve the differential equation $\frac{dy}{dx} + 3x^2y = 6x^2$.

SOLUTION The given equation is linear since it has the form of Equation 1 with $P(x) = 3x^2$ and $Q(x) = 6x^2$. An integrating factor is

$$I(x) = e^{\int 3x^2dx} = e^{x^3}$$

Multiplying both sides of the differential equation by e^{x^3}, we get

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = 6x^2e^{x^3}$$

Or $\qquad \frac{d}{dx}\left(e^{x^3}y\right) = 6x^2e^{x^3}$

Integrating both sides, we have

$$e^{x^3}y = \int 6x^2e^{x^3}\,dx = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}$$

EXAMPLE:

Find the solution of the initial value problem

$$x^2y' + xy = 1 \qquad x > 0 \qquad y(1) = 2$$

SOLUTION:

We must fine first divide both sides by the coefficient of y' to put the differential equation into standard form:

$y' + \frac{1}{x}y = \frac{1}{x^2}$ $x > 0$ $\cdots\cdots\cdots\cdots \rightarrow 1$

The integrating factor is $I(x) = e^{\int\left(\frac{1}{x}dx\right)} = e^{\ln x} = x$

Multiplication of equation 1 by x gives

$$xy' + y = \frac{1}{x}$$

Or $$(xy)' = \frac{1}{x}$$

Then $$xy = \int \frac{1}{x}dx = \ln x + C$$

And so $$y = \frac{\ln x + C}{x}$$

Since $y(1) = 2$, we have

$$2 = \frac{\ln 1 + C}{1} = C$$

Therefore the solution to the initial-value problem is

$$y = \frac{\ln x + 2}{x}$$

2.4 **INTEGRATING FACTOR METHODS**:

Some equations that are not exact may be multiplied by some factor, a function $u(x, y)$ to make them exact.

When this function $u(x, y)$ exists it is called an integrating factor. It will make the following expression valid:

$\frac{\partial\{u.N(x,y)\}}{\partial x} = \frac{\partial\{u.M(x,y)\}}{\partial y}$

There are some special cases:

- $u(x, y) = x^m y^m$
- $u(x, y) = u(x)$ (i.e., u is function only of x)
- $u(x, y) = u(y)$ (i.e., u is a function only of y)

INTEGRATING FACTORS USING $u(x, y) = x^m y^m$:

EXAMPLE:

$(y^2 + 3xy^3)dx + (1 - xy)dy = 0$

Solution: $M = y^2 + 3xy^3$

$$\Rightarrow \frac{\partial M}{\partial y} = 2y + 9xy^2$$

$$N = 1 - xy$$

$$\Rightarrow \frac{\partial N}{\partial x} = -y$$

So it's clear that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

But we can try to make it exact by multiplying each part of the equation by $x^m y^m$:

$$(x^m y^n y^2 + x^m y^n 3xy^3)dx + (x^m y^n - x^m y^n xy)dy = 0$$

Which "simplifies" to:

$$(x^m y^{n+2} + 3x^{m+1} y^{n+3})dx + (x^m y^n - x^{m+1} y^{n+1})dy = 0$$

And now we have:

$$M = x^m y^{n+1} + 3x^{m+1} y^{n+3}$$

$$\Rightarrow \frac{\partial M}{\partial y} = (n+2)x^m y^{n+1} + 3(n+3)x^{m+1} y^{n+2}$$

$$N = x^m y^n - x^{m+1} y^{n+1}$$

$$\Rightarrow \frac{\partial N}{\partial x} = mx^{m-1} y^n - (m+1)x^m y^{n+1}$$

And we want $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So let's choose the right values of m and n to make the equation exact.

Set them equal:

$$(n+2)x^m y^{n+1} + 3(n+3)x^{m+1} y^{n+2} = mx^{m-1} y^2 - (m+1)x^m y^{n+1}$$

Re-order and simplify:

$$[(m+1) + (n+2)]x^m y^{n+1} + 3(n+3)x^{m+1} y^{n+2} - mx^{m-1} y^n = 0$$

For it to be equal to zero, every coefficient must be equal to zero, so:

$$\Rightarrow (m+1) + (n+2) = 0$$

$$\Rightarrow 3(n+3) = 0$$

$$\Rightarrow m = 0$$

That last one, $m = 0$, with $m = 0$ we can figure that $n = -3$

And the result is: $x^m y^n = y^{-3}$

We now know to multiply our original differential equation by y^{-3}:

$$(y^{-3}y^2 + y^{-3}3xy^3)dx + (y^{-3} - y^{-3}xy)dy$$

Which becomes:

$$(y^{-1} + 3x)dx + (y^{-3} - xy^{-2})dy = 0$$

And this new equation should be exact but let's check again:

$$M = y^{-1} + 3x$$

$$\Rightarrow \frac{\partial M}{\partial y} = -y^{-2}$$

$$N = y^{-3} - xy^{-2}$$

$$\Rightarrow \frac{\partial N}{\partial x} = -y^{-2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So, therefore, they are exact.

So, let's continue:

$$I(x, y) = \int N(x, y)dy$$

$$I(x, y) = \int (y^{-3} - xy^{-2})dy$$

$$I(x, y) = \frac{-1}{2}y^{-2} + xy^{-2} + xy^{-1} + g(x)$$

Now, to determine the function $g(x)$ we evaluate

$$\frac{\partial I}{\partial x} = y^{-1} + g'(x)$$

And that equals $M = y^{-1} + 3x$, so

$$y^{-1} + g'(x) = y^{-1} + 3x$$

And so:

$$g'(x) = 3x$$

$$g(x) = \frac{3}{2}x^2$$

So our general solution of $I(x, y) = C$ is:

$$\frac{-1}{2}y^{-2} + xy^{-1} + \frac{3}{2}x^2 = C$$

INTEGRATING FACTORS USING $u(x, y) = u(x)$:

For $u(x, y) = u(x)$ we must check for this important condition:

The expressions $Z(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$

Must not have the y term, so that the integrating factor is only a function of x.

If the above condition is true then our integrating factor is:

$$u(x) = e^{\int Z(x)dx}$$

INTEGRATING FACTORS USING $u(x, y) = u(y)$:

$u(x, y) = u(y)$ is very similar to the previous case $u(x, y) = u(x)$

so, in a similar way, we have:

The expression $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$ must not have the x term in order for the integrating factor to be a function of only y.

And if that condition is true, we call that expression $Z(y)$ and our integrating factor is

$$u(y) = e^{\int Z(y)dy}$$

2.5 EXACT EQUATIONS:

An "exact" equation is where a first-order differential equation like this:

$M(x, y)dx + N(x, y)dy + 0$ has some special function $I(x, y)$ whose partial derivatives can be put in place of M and N like this: $\frac{\partial I}{\partial x}dx + \frac{\partial I}{\partial y}dy = 0$ and our job is to find that magical function $I(x, y)$ if it exists.

We can know at the start if it is an exact equation or not!

Imagine we do these further partial derivatives:

$$\frac{\partial M}{\partial y} = \frac{\partial^2 I}{\partial y \partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 I}{\partial y \partial x}$$

They end up the same! And so this will be true:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

When it is true we have an "exact equation" and we can proceed.

And to discover $I(x, y)$ we do either:

- $I(x, y) = \int M(x, y)dx$ (with x as an independent variable) or
- $I(x, y) = \int N(x, y)dy$ (with y as an independent variable) and then there is some extra work to arrive at the general solution $I(x, y) = C$.

EXAMPLE:

Solve

$$(3x^2y^3 - 5x^4)dx + (y + 3x^3y^2)dy = 0$$

Solution: In this case we have,

- $M(x, y) = 3x^2y^3 - 5x^4$
- $N(x, y) = y + 3x^3y^2$

We evaluate the partial derivatives to check for exactness.

- $\frac{\partial M}{\partial y} = 9x^2y^2$
- $\frac{\partial N}{\partial x} = 9x^2y^2$

They are same! So our equation is exact.

We can proceed.

Now we want to discover $I(x, y)$. Let's do the integration with x as an independent variable: $I(x, y) = \int M(x, y)dx$

$$= \int (3x^2y^3 - 5x^4)dx$$

$= x^3y^3 - x^5 + f(y)$, since $f(y)$ is our version of the constant of integration "c" (due to partial derivative). We had y as a fixed parameter that we know is really a variable. So now we need to discover $f(y)$. At the very start of this page we said $N(x, y)$ can be replaced by $\frac{\partial I}{\partial y}$, so:

$$\frac{\partial I}{\partial y} = N(x, y)$$

Which gets us:

$3x^3y^2 + \frac{df}{dy} = y + 3x^3y^2$

Cancelling terms:

$\frac{df}{dy} = y$

Integrating both sides:

$$f(y) = \frac{y^2}{2} + c_1$$

We have $f(y)$, now just put in the place $I(x, y) = x^3y^3 - x^5 + \frac{y^2}{2} + c_2$ and the general solution is $I(x, y) = C$

So we get $x^3y^3 - x^5 + \frac{y^2}{2} = C\ ' \therefore C = C_1 + C_2$. And that's how this method works.

CHAPTER 3

NUMERICAL SOLUTION TECHNIQUES

3.1 INTRODUCTION:

In many real-world problems, mathematical models are formulated using differential equations, algebraic equations, or systems of equations. Numerical solution techniques provide an essential and practical approach for obtaining approximate solutions with desired accuracy.

Numerical methods involve the use of algorithms to perform step-by-step computations, enabling the solution of mathematical problems that are otherwise unsolvable by traditional analytical methods. Common numerical methods include Euler's method, Runge-kutta methods, finite difference methods, and iterative solvers such as the Newton-Raphson method. These methods are implemented in computational tools and programming environments to analyze and simulate systems efficiently.

3.2 EULER'S METHOD:

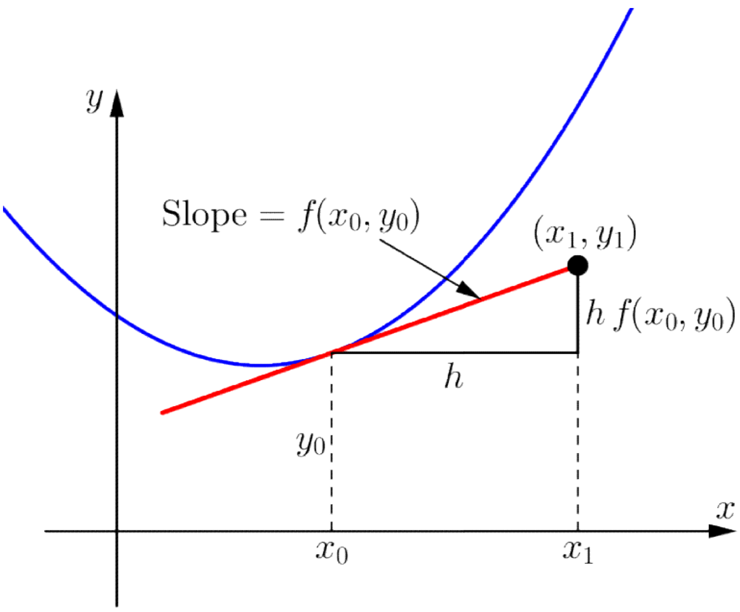
Let the differential equation be

$$\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0 \qquad \cdots\cdots\cdots\cdots \rightarrow 1$$

Integrating (1), we get a relation between y and x which can be written in the form

$$y = F(x) \qquad \cdots\cdots\cdots\cdots \rightarrow 2$$

In xy-plane, the equation (2) represents a curve practically a smooth curve is straight for a short distance from any point on it.



Hence we have the approximate relation $\Delta y \simeq \Delta x tan\theta$

$$(y_1 - y_0) \simeq \triangle x \left(\frac{dy}{dx}\right)_0 = \triangle x f(x_0, y_0)$$

$$\therefore y_1 \simeq y_0 + \triangle x f(x_0, y_0)$$

$y_1 \simeq y_1 + hf(x_0, y_0)$. Then y_1 is the approximate value of y for $x = x_0$, similarly the values of y corresponding to $x_2 = x_1 + h, x_3 = x_2 + h, \dots \dots \dots . . etc.$, are given by

$$y_2 \simeq y_1 + hf(x_1, y_1)$$

$y_3 \simeq y_2 + hf(x_2, y_2)$ etc.,

In general, we obtain $y_{n+1} \simeq y_n + hf(x_n, y_n), n = 0,1,2, \dots \dots . . etc., \dots \dots \dots \dots \dots \dots \rightarrow 3$

Taking h small enough and continuing in this way we could get the integral in a set of corresponding values of x and y.

This process in very slow. For practical use the method, unsuitable because to get reasonable accuracy with this method we need to give a comparatively smaller value to h. If h is not small than the method is too accurate.

EXAMPLE:

Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with $y = 1$ for $x = 0$, find y approximately for $x = 0.1$ by Euler method (five steps).

Solution: Here, $\frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x}$ and $x_0 = 0$, $y_0 = 1$, $i.e., y(0) = 1$. To find out $y(0.1)$ =?

Take $h = 0.02$ for making five steps

Now by Euler formula,

$$y_{n+1=y_n+hf(x_n,y_n)}$$

Now, $y_1 = y_0 + hf(x_0, y_0)$

$$= 1 + 0.02(\frac{y_0 - x_0}{y_0 + x_0})$$

$$= 1 + 0.02(\frac{1 - 0}{1 + 0})$$

$$= 1 + 0.02 \times 1$$

$$= 1.02$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= y_1 + h(\frac{y_1 - x_1}{y_1 + x_1})$$

$$= 1.02 + 0.02 \times \frac{1.02 - 0.02}{1.02 + 0.02}$$

$$= 1.02 + 0.01923$$

$$= 1.03923$$

$$y_3 = y_2 + hf(x_1, y_2)$$

$$= 1.0392 + 0.02(\frac{1.0392 - 0.04}{1.0392 + 0.04})$$

$$= 1.0392 + 0.02(\frac{0.9992}{1.0792})$$

$$= 1.0392 + \frac{0.019984}{1.0792}$$

$$= 1.0392 + 0.01852$$

$$=1.05772$$

$$y_4 = y_3 + hf(x_3, y_3)$$

$$= 1.05772 + 0.02(\frac{1.05772 - 0.06}{1.05772 + 0.06})$$

$$= 1.05772 + 0.02(\frac{0.99772}{1.11772})$$

$$= 1.05772 + \frac{0.01995}{1.11772}$$

$$= 1.05772 + 0.01785$$

$$= 1.07557$$

$$y_5 = y_4 + hf(x_4, y_4)$$

$$= 1.07557 + 0.02(\frac{1.07557 - 0.08}{1.07557 + 0.08})$$

$$= 1.07557 + 0.02(\frac{0.99557}{1.15557})$$

$$= 1.07557 + 0.01723$$

$$= 1.0928$$

ADVANTAGES OF EULER'S METHOD:

The Euler's method has a straightforward algorithm that is easy to code and understand, making it ideal for beginners. It requires minimal computations per step, which can be advantageous for quick approximations or limited-resource environments. When the step size is very small, the method can provide reasonably accurate results for smooth functions. It serves as the basis for understanding more complex numerical methods like Runge-Kutta and Predictor-Corrector methods. It can provide insights into the qualitative behavior of a differential equation's solution without requiring a full analytical solution. It can be applied to a wide variety of first-order ODEs and can also be extended to systems of ODEs.

DISADVANTAGES OF EULER'S METHOD:

In Euler's method $\frac{dy}{dx}$ changes rapidly over an interval; this gives a poor approximation at the beginning of the process in comparison with the average value over the interval. So the calculated value of y in this method occurs much error than the exact value, which reasonably increased in the succeeding intervals, then the final value of y differs on a large scale from the exact value.

Euler's method needs to take a smaller value of h, because of this restriction this method is unsuitable for practical use and can be applied for tabulating the value of depending variable over a limited range only. Moreover, if h is not small enough this method is too inaccurate. In Euler's method, the actual solution curve is approximated by the sequence of short straight lines, which sometimes deviates from the solution curve significantly.

3.2 TAYLOR'S SERIES METHOD:

Let us consider the initial value problem

$$y' = \frac{dy}{dx} = f(x, y); \qquad y(x_0) = y_0 \qquad \ldots\ldots\ldots 1$$

Let $y = y(x)$ be the exact solution of (1) such that $y(x_0) \neq 0$. Now expanding (1) by Taylor's series about the point $x = x_0$, we get $y = y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' \qquad \cdots\cdots\cdots \rightarrow 2$

In the expression (2), the derivatives $y_0', y_0'', y_0''', \ldots\ldots\ldots$ are not explicitly known. However, if $f(x, y)$ is differentiable several times, the following expression in terms of $f(x, y)$ and its partial derivatives as the followings

$$y' = f(x, y) = f$$

$$y'' = f'(x, y) = f_x + y' f_y = f_x + f f_y$$

$$y''' = f''(x, y) = f_{xx} + 2f f_{xy} + f_{yy} f^2 + f_x f_y + f_y^2 f$$

In a similar manner, a derivative of any order of y can be expressed in terms of $f(x, y)$ and its partial derivatives.

As the equation of higher-order total derivatives creates a hard stage of computation, to overcome the problem we are to truncate Taylor's expansion to the first few convenient terms of the series. This truncation in the series leads to a restriction on the value of x for which expansion is a approximation.

Now, for suitable small step length $h = x_i - x_{i-1}$, the function $y = y(x)$ is evaluated at $x_1 = x_0 + h$. Then the Taylor's expansion (2) becomes

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots \dots \dots \dots$$

Or, $y_1 = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0''' + \cdots \dots \dots \dots \quad \dots \dots \dots \dots \rightarrow 3$

The derivatives y_0', y_0'', y_0''', … … … … … are evaluated at $x_1 = x_0 + h$, and then substituted in (3) to obtain the value of y at $x_2 = x_0 + h$ given by

$$y(x_0 + 2h) = y(x_0 + h) + hy'(x_0 + h) + \frac{h^2}{2!} y''(x_0 + h) + \frac{h^3}{3!} y'''(x_0 + h) + \cdots \dots \dots \dots$$

$$or, y_2 = y_1 + hy_1' + \frac{h^2}{2} y_1'' + \frac{h^3}{6} y_1''' + \cdots \dots \dots \dots \dots \quad \dots \dots \dots \dots \dots \rightarrow 4$$

By similar manner we get

$$y_3 = y_2 + hy_2' + \frac{h^2}{2} y_2'' + \frac{h^3}{6} y_2''' + \cdots \dots \dots \dots$$

$$y_4 = y_3 + hy_3' + \frac{h^2}{2} y_3''' + \cdots \dots \dots \dots \dots$$

Thus the general form obtained as

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + \cdots \dots \dots \dots \dots \dots \dots \dots \rightarrow 5$$

This equation can be used to obtain the value of y_{n+1}, which is the approximate value to the actual value of $y = y(x)$ at the value $x_{n+1} = x_0 + (n+1)h$.

EXAMPLE:

Apply Taylor's series method to solve $\frac{dy}{dx} = x - y$ with the initial condition $y(0) = 1$ up-to $x = 0.4$ where $h = 0.2$.

Solution: Given that $y' = \frac{dy}{dx} = x - y = f(x, y)$

Also $y_0 = 1$ when $x_0 = 0$ and $h = 0.2$, so that $y' = x - y$

$$y'' = 1 - y'$$

$$y''' = -y''$$

$$y'^{v} = -y'''$$

$$y^{v} = -y'^{v}$$

$$y^{v'} = -y^{v}$$

To find y_1 we are to proceed as follows:

$$y_0' = x_0 - y_0 = -1$$

$$y_0'' = 1 - y_0' = 2$$

$$y_0''' = -y_0'' = -2$$

$$y_0'^{v} = -y_0''' = 2$$

$$y_0^{v} = -y_0'^{v} = -2$$

$$y_0^{v'} = -y_0^{v} = 2$$

Now from equation (3) we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0''' + \frac{h^4}{24} y_0'^{v} + \frac{h^5}{120} y_0^{v} + \frac{h^6}{720} y_0^{v'} + \cdots \ldots$$

Neglecting the terms containing h^7 and higher-order terms and by substituting the values of $y_0', y_0'', y_0''', y_0'^{v}, y_0^{v}$ & $y_0^{v'}$, we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0'^v + \frac{h^5}{120}y_0^v + \frac{h^6}{720}y_0^{v'}$$

$$= 1 + (0.2)(-1) + \frac{(0.2)^2}{2}(2) + \frac{(0.2)^3}{6}(-2) + \frac{(0.2)^4}{24}(2) + \frac{(0.2)^5}{120}(-2) + \frac{(0.2)^6}{720}(2)$$

$$= 0.837461511\ (app)$$

i.e. $y_1 = 0.837461511$ & $x_1 = x_0 + h = 0.0 + 0.2 = 0.2$

To find y_2 we are to proceed as follows:

$$y_1' = x_1 - y_1 = -0.637461511$$

$$y_1'' = 1 - y_1' = 1.637461511$$

$$y_1''' = -y'' = -1.637461511$$

$$y_1'^v = -y_1''' = 1.637461511$$

$$y_1^v = -y_1'^v = -1.637461511$$

$$y_1^{v'} = -y_1^v = 1.637461511$$

$$y_2 = y_1 + hy_1' + \frac{h^2}{2}y_1'' + \frac{h^3}{6}y_1''' + \frac{h^4}{24}y_1'^v + \frac{h^5}{120}y_1^v + \frac{h^6}{720}y_1^{v'} + \cdots \ldots \ldots$$

Neglecting the terms containing h^7 and higher-order terms and by substituting the values of $y_1', y_1'', y_1''', y_1'^v, y_1^v$ & $y_1^{v'}$, we get

$$y_2 = y_1 + hy_1' + \frac{h^2}{2}y_1'' + \frac{h^3}{6}y_1''' + \frac{h^4}{24}y_1'^v + \frac{h^5}{120}y_1^v + \frac{h^6}{720}y_1^{v'}$$

$$= 1 + (0.2)(-1.637461511) + \frac{(0.2)^2}{2}(1.637461511) + \frac{(0.2)^3}{6}(-1.637461511) + \frac{(0.2)^4}{24}(1.637461511)$$
$$+ \frac{(0.2)^5}{120}(-1.637461511) + \frac{(0.2)^6}{720}(1.637461511)$$

$$= 0.740640099(app)$$

i.e. $y_2 = 0.740640099$ & $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

Thus we get $y_1 = 0.837461511$ & $y_2 = 0.740640099$

CHAPTER 4.

REAL LIFE APPLICATIONS

1.4 INTRODUCTION:

First-order ordinary differential equations (ODEs) are fundamental mathematical tools used to describe various dynamic systems that evolve over time. These equations involve a function and its first derivative, representing the rate of change of a quantity with respect to another, typically time or space. Due to their simplicity and effectiveness, first-order ODEs are widely applied in modeling real-life phenomena across diverse fields.

In science and engineering, first-order ODEs model the behavior of systems where the current rate of change depends solely on the presents state. For example, in physics, they describe exponential decay and growth processes such as radioactive decay and cooling of objects (Newton's Law of Cooling). In biology, they help model population growth and drug concentration in the bloodstream. In economics, they are used to represent systems involving interest rates and investment growth. Additionally, first-order ODEs find :resistors and capacitors (RC circuits).

This chapter explores the practical significance of first-order ODEs by highlighting key examples and case studies where these equations are used to solve real-world problems. By understanding these applications, we gain insight into how mathematical concepts are used to model and predict natural and engineered processes, thereby bridging the gap between theoretical mathematics are real-life situations.

1.5 REAL-WORLD PROBLEMS MODELED USING FIRST-ORDER ODEs:

These are numerous real life applications for first order differential equations to real life systems. In this study we shall discuss the following

- Population growth and decay
- Newton's law of cooling

POLUTION GROWTH AND DECAY:

Population growth involves a dynamic process which can be developed using differential equation. The exponential growth model or natural growth model is known as Mathus' model. This model is based on the assumption that the rate of change of the population is proportional to the existing population itself. If $p(t)$ represents the total population at time t, the above assumption can be written as

$$\frac{dp}{dt} = kp(t) \dots \dots \dots \dots (1)$$

Where k is the proportionality constant.

The above model (1) can also be used in financial institute for example, when saving money in the bank, the balance in savings account with interest compounded continuously exhibits natural growth provided no withdrawal and in this case the constant k represents the annual rate of interest, group of animal population grows exponential provided size is not affected by environmental factors, in this case k is known as the productivity rate of population and it can used in migration.

To obtain solution of (1) we multiply the equation with e^{-kt}, the integrating factor

$$e^{-kt}\frac{dp}{dt} = kpe^{-kt}$$

$$e^{-kt}\frac{dp}{dt} - kpe^{-kt} = 0$$

$$\frac{dp}{dt}[pe^{-kt}] = 0$$

$$\int \frac{dp}{dt}[pe^{-kt}] = \int 0$$

$$pe^{-kt} = C \; or \; p - ce^{kt}$$

Suppose the initial population is p_0 then

$$p(0) = p_0 \; and \; C = p_0$$

$$p(t) = p_0 e^{kt} \dots \dots \dots \dots (2)$$

When $k > 0$, the population grows and when $k < 0$, the population decays.

EXAMPLE:

Suppose the population of a certain community is known to increase at a rate proportional to the number of people living in the community at time t, the population has doubled after 7years, how long would it take to triple? If it is known that the population of the community is 12,000 after 5years, determine the initial population and predict the population in 40years.

SOLUTION: Let p_0 denotes initial population of the community and $p(t)$ the population of the community at any time t, then we have

$\frac{dp}{dt} = kp$

$p(t) = p_0 e^{kt}$

Given that

$p(7) = 2p_0$

$e^{7k} = 2$

$k = \frac{0.6931}{7} = 0.0990$

The solution of the model becomes $p(t) = p_0 e^{0.0990t}$

$$e^{0.0990t} = 3$$

$$0.0990t = \ln 3$$

$$t = \frac{1.0986}{0.0990} = 11.0970 \approx 11 years$$

Apply $p(5) = 12{,}000$

$12{,}000 = p_0 e^{0.0990 \times 5}$

$$p_0 = \frac{12{,}000}{e^{0.450}} = 7{,}315$$

Hence the initial population of the community was $p_0 \approx 7{,}315$. Therefore, solution of the model is $p(t) = 7315e^{0.0990}$

So that the population in 40years is

$$p(40) = 7315e^{40(0.0990)}$$

$$p(40) = 7315e^{3.960}$$

$$p(40) = 7315(52.4573)$$

$$p(40) \approx 383{,}725$$

NEWTON'S LAW OF COOLING:

Another important real life application of differential equation in Newton's law of cooling Sir Isaac Newton developed huge interest in quantitative finding the loss of heat in a body and a formula was derived to represent this phenomenon. The law states that the rate of change of temperature of a body is directly proportional to the difference is solid object and surrounding environment at a given instant of time

$$\frac{dT}{dt} \propto (T_0 - T_5)$$

$$\frac{dT}{dt} = k(T_0 - T_5)$$

Where $T(0) = T_0 \dots \dots \dots . . (1)$

T_0= Temperature of the body

T_5 = Temperature of surrounding

k = Constant of proportionality

$$\int \frac{dT}{T_0 - T_5} = \int kdt$$

$$\ln|T_0 - T_5| = kt + C$$

$$T_0 - T_5 = e^{kt+c}$$

Applying $T(0) = T_0$ yields

$$C = T_0 - T_5$$

$$T(t) = T_5 + (T_0 - T_5)e^{kt}$$

Suppose the temperatures at t_1 and t_2 are given we have

$$T(t_1) - T_5 = (T_0 - T_5)e^{kt_1}$$

$$T(t_2) - T_5 = (T_0 - T_5)e^{kt_2}$$

So that

$$\frac{T(t_1) - T_5}{T(t_1) - T_5} = e^{k(t_1 - t_2)} \ldots\ldots\ldots\ldots\ldots \rightarrow (2)$$

EXAMPLE:

A police man discovered a dead body at the midnight in a room when the temperature of the dead is 90^0F, the body temperature of the room was kept constant at 70^0F. Three hours after the temperature of the body dropped to 85^0F. Determine the time of death of the victim .

Solution:

$T(0) = 98.6^0F \quad (37^0C) = T_0$ and $T_5 = 70^0F$

Provided the victim was not sick

$\frac{dT}{dt}k(T_0 - 70), \quad T(0) = 98.6$

But $T(t) = T_5 + (T_0 - T_5)e^{kt}$

So that

$$\frac{T(t_1) - T_5}{T(t_2) - T_5} = e^{k(t_1 - t_2)}$$

$T(t_1) = 90^0F$ and $T(t_2) = 85^0F$

$$\frac{90 - 70}{85 - 70} = e^{3k}$$

$$t_1 - t_2 = 3\ hours$$

$$k = \frac{1}{3}\ln\frac{4}{3} = 0.0959$$

Let t_1 and t_2 represent the times of death and discovery of the dead body then

$$T(t_1) = T(0) = 98.6^0F$$

And $T(t_2) = 90^0F$

The time of death $(t_3) = t_1 - t_2$ and we have

$\frac{T(t_1)-T_5}{T(t_2)-T_5} = e^{kt_3}$

$\frac{98.6-70}{90-70} = e^{kt_3}$

$t_3 = \frac{1}{k}\ln\frac{28.6}{20} = \frac{1}{0.0959}\ln\frac{28.6}{20} \approx 3.730$

Therefore, the person died at about 8:18 pm.

4.3DISCUSSION OF MODEL ASSUMPTIONS AND LIMITATIONS:

ASSUMPTIONS IN FIRST-ORDER ODE MODELS:

First-order ordinary differential equations are often used to model systems where the rate of change of a quantity depends only on the current state of that quantity and time. The following are typical assumptions made:

- **Deterministic System:** The system is assumed to behave in a predictable way; randomness or uncertainty is not included in the model.
- **Continuity and Smoothness:** The functions involved (e.g., the derivative and any external input functions) are continuous and differentiable.
- **Time Invariant or Time Dependent Only:** The rate of change is typically a function of the dependent variable and time only: $\frac{dy}{dt} = f(t, y)$. This assumes no dependence on other complex variables unless explicitly modeled.
- **Initial Condition Known:** An initial condition (e.g., $y(t_0) = y_0$) is known, allowing for the unique solution of the ODE.
- **Local Linearity (in linear models):** Many models assume a linear relationship (e.g., exponential growth or decay), which may not be valid for all real-world scenarios.

LIMITATIONS OF FIRST-ORDER ODE MODELS:

While useful, first-order ODEs come with several important limitations:

- **Limited Scope of Dynamics:** First-order ODEs can only capture systems where the behavior is governed by a single rate of change. They cannot model systems with acceleration or inertia, which require second or higher-order equations.
- **Simplistic Representations:** Many physical, biological, and economic systems are more complex than what a first-order ODE can capture. Such models may overlook feedback loops, delays, or non-local interactions.

- **No Stochastic Behavior:** First-order ODEs do not account for randomness or uncertainty (as opposed to stochastic differential equations, or SDEs).
- **Sensitivity to Initial Conditions:** Some models may be highly sensitive to small errors in the initial condition, especially in nonlinear systems.
- **Idealized Inputs:** The influence of external forces or inputs is often simplified (e.g., constant or sinusoidal), which may not match real-world variability.

EXAMPLES OF APPLICATION AND LIMITATION:

- **Population Growth:** Modeled as $\frac{dp}{dt} = rP,$ assumes unlimited resources and no migration-not realistic in the long term.
- **Newton's Law of Cooling:** $\frac{dT}{dt} = -k(T - T_{env})$, assumes constant ambient temperature and no phase change, which limits its accuracy.

CONCLUSION: First-order ODEs are powerful tools for modeling simple dynamic systems, especially when the rate of change depends on only one variable. However, they often rely on idealized assumptions and may not be sufficient for modeling complex or multi-variable systems. Awareness of these assumptions and limitations is crucial for accurately interpreting the results and refining models when necessary.

CHAPTER 5.

CONCLUSION

5.1 SUMMARY OF KEY FINDINGS:

Classification of First-Order ODEs:

- Identified and explained various types: separable, linear, exact, and homogeneous equations.
- Presented methods for recognizing and transforming equations to suitable forms for easier solving.

Analytical Solution Techniques:

- Demonstrated methods such as separation of variables, integrating factor, substitution, and exact equation techniques.
- Showed how each method applies under specific conditions and highlighted their effectiveness.

Qualitative Analysis:

- Explored direction fields and isoclines to visualize solution behaviors without solving the equations.
- Identified stability and equilibrium points for different differential models.

Applications in Real-World Problems:

- Modeled physical, biological, and economic phenomena like population growth, Newton's cooling law, and simple electrical circuits.
- Validated models using real or simulated data, showing the relevance of ODEs in practical scenarios.

Numerical Methods and Limitations:

- Applied Euler's method and Runge-kutta methods for cases where analytical solutions are not feasible.
- Discussed error estimation and convergence of numerical approaches.

Theoretical Contributions:

- Provided proofs for existence and uniqueness theorems (e.g., Picard-Lindelof theorem).
- Analyzed conditions under which solutions behave continuously with respect to initial conditions.

Limitations and Challenges:

- Noted the difficulties in solving non-linear and non-exact equations analytically.
- Identified potential gaps in numerical precision and stability for stiff equations.

Recommendations for Further Study:

- Suggested extending analysis to systems of first-order ODEs or higher-order equations.
- Encouraged incorporating machine learning and symbolic computation tools for solving and visualizing complex models.

5.2 SIGNIFICANCE OF FIRST-ORDER ODES IN MODERN SCIENCE:

The **significance of first-order ordinary differential equations (ODEs)** in modern science is vast and foundational. These equations model how quantities change over time or space and are critical in understanding dynamic systems. Here are key areas highlighting their importance:

1) **Modeling Natural Phenomena:** First-order ODEs are essential tools in describing and predicting real-world processes, such as:
   - Population dynamics (e.g., exponential growth or decay)
   - Radioactive decay in nuclear physics
   - Cooling and heating laws (Newton's Law of Cooling)
   - Motion of objects under velocity-dependent forces
2) **Applications in Physics and Engineering**
   - In electric circuits: They describe voltage and current relationships using Kirchhoff's laws.
   - In mechanical systems: Damping and velocity-dependent forces are modeled using first-order ODEs.
   - In fluid mechanics: Flow rates and diffusion processes often involve first-order relations.

3) **Biology and Medicine**
    - Pharmacokinetics: Drug absorption and elimination in the body follow first-order kinetics.
    - Epidemiology: The rate of spread of infectious diseases (e.g., SIR models) often starts with first-order differential equations.
4) **Economics and Social Science**
    - Modeling interest rates, investment growth, and economic decay often involves first-order ODEs.
    - They are used in game theory and dynamic systems analysis in sociology and psychology.
5) **Computational Simulations and Control Systems**
    - First-order ODEs are frequently used in numerical simulations for robotics, climate modeling, and automated systems.
    - Control theory employs them in designing systems that maintain desired outputs.
6) **Foundation for Higher-Order Models**
    - Many complex systems (like second or higher-order ODEs and partial differential equations) are built upon first-order ODE principles.
    - They provide a basis for qualitative analysis, such as phase portraits and stability of equilibria.

CONCLUSION

First-order ODEs are not only mathematically elegant but also immensely practical. They form the building blocks for modeling, analyzing, and solving countless problems across modern science and technology. Their simplicity, combined with their power to describe dynamic change, makes them indispensable in both theoretical and applied sciences.

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