“On Learning Fundamental Concepts of Group Theory”

Dissertation submitted to the Department of Mathematics in fulfillment of the requirement for the award of the degree of Master of Science

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Certificate

This is to certify that the dissertation entitled "ON LEARNING FUNDAMENTAL CONCEPTS OF GROUP THEORY "submitted by Ariful Islam Roll No. MAT-12/23, Registration No. MSSV-0023-101- 001331, in partial fulfillment for M.Sc in Mathematics, is a bonafide record of original work carried out under my supervision and guidance.

To the best of my knowledge, the work has not been submitted earlier to any other institution for the award of any degree or diploma.

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DECLARATION

I, Ariful Islam, hereby declare that the dissertation titled "ON LEARNING FUNDAMENTAL CONCEPTS OF GROUP THEORY "submitted to the Department of Mathematics, Mahapurusha Srimanta Sankaradeva Viswavidyalaya, is a record of original work carried out by me under the supervision of Dr. Raju Bordoloi, HOD

This work has not been submitted earlier to any other institution or university for the award of any degree or diploma.

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Acknowledgment

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Table of contents

Introduction

We're writing this to spark a conversation about how people grasp abstract algebra, particularly group theory. We're curious about the learning process involved in mastering these concepts. By examining the challenges students face with abstract ideas, we aim to contribute to our understanding of human cognition and ultimately enhance math education.

Currently, our focus is on how a group of high school math teachers engaged with group theory. We're identifying common difficulties they encountered and exploring more effective teaching approaches. This is part of a larger, ongoing project centered on the learning and teaching of college-level mathematics.

In the end, we'll briefly share some teaching strategies that emerged from our observations. However, putting these ideas to the test to evaluate their effectiveness will have to wait for another time. Still, we hope to get the discussion started with this piece and continue to delve deeper into this important subject.

What's the Point of Group Theory?

Abstract algebra, especially group theory, poses a real challenge in education. Both professors and students often view it as one of the toughest undergrad courses out there. It seems to trip up students, not just with the material itself but also in how it shapes their feelings about abstract math. Some studies back this up, like those by Hart and Selden & Selden.

In many schools, abstract algebra is the first course where students can't just rely on copying how to solve similar problems. They actually have to grapple with abstract ideas, understand core math principles, and learn to write proofs. Anecdotally, many students say this class makes them want to avoid abstract math altogether. Since a lot of students taking this course are future math teachers, it's really important for math educators to figure out better ways to teach abstraction so that high school teachers feel more comfortable with it.

There's another key reason why abstract algebra—and specifically quotient groups—are so important, and it has to do with abstraction. To truly grasp the concept of a group, a person's understanding should go beyond specific examples. It should include recognizing various mathematical properties and structures that apply generally, even to groups made up of vague elements with a binary operation that simply meets the basic rules.

Even if you start with a very tangible, concrete group, moving to one of its quotient groups changes the game. It alters the nature of the elements involved. This shift forces students to work with elements—like cosets—that feel unfamiliar or undefined to them. This link between abstract groups and their quotient counterparts has significant historical roots too (Nicholson, 1993)[1].

Objectives of This Study

We're guided by two key questions in our ongoing research: How can someone learn specific concepts in elementary group theory? And how does mastering these concepts relate to understanding math and abstract thinking more broadly?

Focusing on Learning Specific Topics

When we examine how individuals tackle particular topics, our aim is to see if we can map out a developmental pathway or a detailed breakdown of the learning process for various mathematical subjects.

Our long-term goal isn't just to outline a sequence of mathematical ideas, but also to identify the specific mental steps or constructions people might use to grasp a particular mathematical concept. We believe that explaining these mental processes will eventually lead us to effective teaching methods for individual topics.

The topics we chose primarily reflect what seems like a sensible way to introduce students to abstract algebra. For decades, curriculum designers have felt that even younger students can handle the fundamental idea of a group and its subgroups, along with the necessary background knowledge of sets and functions (like binary operations) that underpin these concepts. Since this appears to be a logical starting point for an abstract algebra course, our study begins by exploring how people understand these topics and how that understanding develops in their minds.

These ideas often start to take shape in a person's mind . It seems like the biggest hurdles in grasping group theory tend to kick in with the concepts that pave the way to Lagrange's theorem, as well as topics like quotient groups, cosets, coset multiplication, and normality. Because of this, we decided to focus our current investigation on these specific areas.

Making Sense of Math: A Theoretical View

When we look closely at how people try to grasp specific ideas in math, we hope it can give us some insight into the bigger picture of how learning happens. Just how intricate are the ideas around groups, for instance, in the mind of someone who's just starting to understand them? And how can we make sense of and tackle the challenges that pop up along the way?

To tackle these kinds of questions, a key tool we use in this study is the theoretical viewpoint we bring to our exploration. Everything from the questions we ask, the methods we use to ask them, to how we interpret students' answers, all stem from a constructivist approach. This approach, inspired by Piaget's ideas (like in his 1975 work), is something we're adapting to understand advanced mathematical thinking better.

Our work is guided by a strong belief in blending theory, practice, and teaching methods together. This idea echoes the recommendations of educational thinkers like Erickson (1986) and Glaser (1968). Erickson, in particular, champions a method that weaves together theory and data to form "assertions" rather than "proofs" in the typical numerical sense. The qualitative methods we're using to observe our participants aren't about reaching definitive conclusions. Instead, they're about creating interpretations that come from blending theory with real-world practice.

Many theories explain how people learn, and these have been applied to college-level math. Thinkers like Bachelard (1938) and Sierpinska (1992) talked about 'epistemological obstacles'—basically, roadblocks to understanding. Others, like Vinner (1983), focused on the difference between how we define a concept and the mental picture we have of it. Kaput (1987) highlighted the importance of looking at ideas in many different ways (multiple representations), and Sfard (1992) contrasted 'operational' understanding (how things work) with 'structural' understanding (what things are made of). As far as we know, none of these specific ideas have been used to study how students learn abstract algebra.

Our approach tries to take Piaget's idea of 'reflective abstraction' (Dubinsky, 1991) and apply it to college math. While it's quite different, it also shares a lot in common with these other theories, especially Sfard's.

The core idea behind our view is this: when someone encounters a problem that throws their thinking off balance (disequilibration) within a social setting, they naturally try to get back in balance. They might do this by fitting the situation into mental frameworks (schemas) they already have. If that doesn't work, they use reflective abstraction to build more advanced schemas. As we developed this idea, we looked closely at the mental steps involved in this process. We found they generally fall into four main categories: actions, processes, objects, and schemas .An 'action' is simply any physical or mental step you can repeat that changes something in some way.

Think of it this way: when an action can happen entirely within someone's mind, or just be imagined happening, without needing to follow every single step, we call that action "interiorized" and it becomes a process. Once something is a process, the student can use it to create new processes. For instance, they might combine it with other processes, linking the "inputs" and "outputs" together in a way that forms a new process altogether. Another way to get a new process is to simply reverse the original one. And when a process can be changed by some action, then we say it has been "encapsulated" and turned into an object.

One interesting result of this way of looking at things is that symbols and other ways of representing things aren't as important as the mental steps people take to understand and build abstract ideas. We believe that when students struggle with symbols, it's often because they're trying to use labels too early—before they've fully formed the idea into an object in their mind. Once that object exists mentally, naming it is usually straightforward. But understanding what a symbol means requires jumping back from the object to the process it came from. There are many situations in math where it's really important to be able to make that shift—moving back and forth between thinking of something as an object and thinking of it as a process.

One key idea in our theory is that the only way to do this is by 'de-encapsulating' the object. This means revisiting the process that was originally used to create the object.

A collection of related processes and objects can be gathered and organized into what we call a 'schema.' From our theoretical viewpoint, schemas are essentially the forms that concepts take in a person's mind. To tackle a problem, a schema can be 'unpacked' so that the individual processes and objects within it can be worked with. Furthermore, a schema itself can be treated as an object, meaning processes and actions can be applied to it as well.

It's crucial to understand that this analysis isn't separate from having knowledge about a specific subject area. As a person works to re-establish balance, internalize ideas, encapsulate and de-encapsulate concepts, they must also build up a significant amount of knowledge related to that subject. This knowledge building can happen at different speeds depending on the topic. For more detailed discussion on our overall theory and some examples of its application to other mathematical subjects, readers should look at the works by Ayers et al, 1988, Breidenbach et al, 1991, Dubinsky, 1991, and Dubinsky & Harel, 1992.

For our long-term project, a really important question is just how useful these ideas are for understanding the data we collect and for suggesting effective teaching methods. The current work isn't meant to draw any final conclusions about the teaching strategies that might come from our observations. Instead, we're simply offering some thoughts and raising key questions related to the development from actions to processes to objects. We also suspect that perhaps a broader curriculum strategy might be needed, one that moves away from the usual step-by-step approach.

Getting a Feel for How People Learn About Groups

The insights we're talking about came from watching 24 high school math teachers. These teachers were part of a summer program that focused on new, often tech-savvy, ways to teach math. As part of this program, they took a course on abstract algebra. Some of them had studied groups before in college, but for many, that was a long time ago, and it didn't seem like much of that knowledge stuck with them. So, essentially, we were watching a group tackle the concept of groups with limited prior experience. The course they took used written materials and worksheets, which later evolved into a textbook (Dubinsky and Leron, 1993). The sequence of topics followed a fairly common path. We'll dive into some specifics in the "Interpretations" section to explore how the order in which they learned things might have influenced their understanding.

Towards the end of the workshop, the participants were given a written assessment (details in Appendix I) to gauge their grasp of the course material. They had two hours to complete it. The organizers framed this as an "observation" aimed at helping the instructors understand the teachers' learning process, and it's worth noting that no grades were given for the course.

They had two hours to tackle the questions. We explained to everyone that this was simply an "observation" meant to help the instructors get a sense of their learning progress, and emphasized that no grades were involved.

After reviewing the test results, we chose 10 individuals and asked them to sit down for recorded interviews to discuss their answers. We deliberately selected participants who had answered questions correctly, partially correctly, and incorrectly, hoping to capture a variety of perspectives on the material. We also leaned towards interviewing those who seemed to be actively working through the concepts, rather than just those who clearly understood everything or those who clearly didn't get it at all.

During these sessions, institute faculty members asked the questions listed in Appendix II and gently probed for insights that might not have shown up on the written test. All of these interviews took place before we ever mentioned the test results to the larger group in the workshop.

We organized the written notes from the participants into different groups based on specific topics from group theory. Within each group, we used our theoretical viewpoint to examine how the participants seemed to think about these topics. We believe that the participants' answers align nicely with the theoretical perspective we've outlined and help show some of the challenges they might be facing. However, considering the limited amount of data and the complex nature of the mathematical concepts, we're not making any claims that this study has validated or confirmed the theory. Instead, we're presenting the following discussion as one possible way to understand how these students might have been learning about groups.

Here are some thoughts we believe offer plausible interpretations of the experiences of our high school teacher participants, presented in a style similar to Erickson's "assertions." We consider their responses to be typical of students grappling with these concepts, but we're sharing their experiences as examples rather than as definitive proof of our theory. Taken together, these examples and the theory provide an initial interpretation of the written assessments and interviews, and they seem to us to raise some intriguing questions that we and others can explore further in ongoing research.

<u>Developing the Concepts of Group and Subgroup.</u>

To make our discussion clearer and hopefully offer some deeper insights, we've organized our thoughts into two main parts. The first part focuses on how our students might have been developing a general idea of groups, starting from a specific example, and grasping the concept of a subgroup. The second part explores a possible progression in understanding the mathematical ideas that lead up to the concept of a quotient group, including the notion of a coset, how one might mentally work with operations on cosets, and the idea of normality.

Let's talk about how people learn the ideas of "group" and "subgroup." We think someone can start putting these two concepts together at the same time. Our observations suggest that understanding these ideas often happens gradually, moving through various stages where the grasp isn't yet complete. This understanding can start by seeing groups and subgroups mainly as collections of separate items. Then, it progresses to a stage where not just the items but also the operations that work on them become part of the core definition. Finally, a student might build a solid understanding of a group as a single entity—an object that actions can be performed upon.

<u>Getting to Grips with Groups and Subgroups as Sets of Elements</u>.

It seems that some students tackle problems involving a set and an operation by relying on an existing mental framework for sets, while overlooking the operation that's also part of the scenario. We believe this approach might point to an early misunderstanding of the ideas behind groups and subgroups.

When first learning about groups, a student might think of a group mainly in terms of its individual elements, basically seeing it as a collection or set. If someone stays at this basic level of understanding, they might only be able to tell one group apart from another by the number of elements it contains.

One example of a student's response which may indicate a strong emphasis on groups as sets of elements occurred when Kim was asked if z_6 were isomorphic toS_3?[2] Kim says the following3

2z_3 is the group of elements {0, 1, 2} with the operation of addition modulo 3; $z_6$is the group of elements {0, 1, 2, 3, 4, 5} with the operation of addition modulo 6; in general, Zn, is the group of elements {0,1,..., n} with the operation of addition modulo n.

$s_3$is the symmetric group on {1, 2, 3}, ie., the group of all permutations of 3 objects. In general, s_n, is the group of all permutations of n objects.

3 Names of students have been changed to avoid identification of individuals. Kim: Probably so, $s_3$has 6 elements in it and z_n has 6 elements in it, so without going through the whole procedure, I would say yes.

In addition to confusion about isomorphism, this student's understanding seems to emphasize the number of elements as a characterizing feature of a group.

Thus, it may be that $z_3$is considered to be any set with three elements that is known to be a group. For example, in the written assessment and the interview, another student, Cal, variously considers $z_3$to be the set {0, 1, 2}, {1, 2, 3}, {0, 2, 3}, or {0,2,4}.

Also consider Sue who answered Question 1(b) (on subgroups of z_6) on the written assessment, (Appendix I), specifying a group by its elements; she wrote {10} for a subgroup of z_6 with two elements and {2 1 0} for a subgroup of z_6 with three elements.

At the earliest stages of understanding groups, the students may construct their own idea of group by considering familiar objects (elements of the group) and forming a process of associating these objects with each other in a set. Eventually, the students may encapsulate that process into an object which, for them, represents the group in question.

<u>Subgroup as a subset</u>

Understanding a subgroup as a subset is similar to understanding a group as a set. For a student at this stage, sometimes "being a subset", that is, having all its elements included in a bigger set, is sufficient to conclude the existence of a subgroup. In other cases students require that such subsets of elements share a common property.

In looking for subgroups of d_3 , many students correctly mentioned the "rotations". Similarly, but incorrectly, some listed "the flips" as a subgroup. Consider for example Cal who, in responding to Question 2(a) of the written assessment (Appendix I), listed the elements of D_3 as $\{ R_0 , R_1, R_2, D_1, D_2, D_3\}$ and identified the first three as the rotations and the second three as the flips. Then in responding to Question 2(c) he listed $\{R_0, R_1,R_3 \}$ as a subgroup of D_3 isomorphic to Z_3 and in responding to Question 2(d) he listed $\{D_1, D_2, D_3\}$ as a subgroup of D_3 also isomorphic toZ_3. In all cases, he mentions the correct operation. Here is what happens when the interviewer asks Cal about his choice of {D$_1$, D_2, $D_3\}$ as a subgroup

I: And what about this one here? You want it isomorphic to Z3. What you write here. is $\{D_1, D_2, D_3\}$.

Cal: Yeah, I thought if you do them all....

I: The three flips.

Cal: Right.

I: You think it's a subgroup. $4D_3$ is the group of symmetries of a triangle. In general, D., is the group of all symmetries of a regular n-sided polygon.

The designation I: refers to the interviewer in all of the interview excerpts in this paper, although it is not necessarily the same person each time.

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Cal: Well, like you told me you have to have the same operation, it works on it the same as addition.

I: Well, that's not the point because it has to be a subgroup of this D_3. But is it a group at all under composition?

Cal: I thought it was. I didn't see anything that...I thought it was closed.

The interviewer then prompts Cal to see that the group properties are not satisfied by this subset. Thus we see that Cal appears to understand a subgroup as any coherent subset. The group operation is carried along, but it is not used.

Individuals who have not progressed beyond this point would probably have no difficulty in considering the even integers to be a subgroup of Z, but they might also think that the odd integers were a subgroup as well.6

This demonstrates a misconception caused by some students' efforts to construct a new concept (group) by relating it to a familiar concept (set). This is an example of requilibration by assimilating the situation to existing available schemas before those schemas have been reconstructed to achieve a higher level of sophistication. It may happen that a student leaps over this step, or passes through it very quickly. But nevertheless, as we witnessed above, some students exhibited vestiges of this mis- conception after five weeks (approximately 50 contact hours) of instruction in group theory.

Seeing Groups and Subgroups as Sets with Operations.

As students encounter situations in which their current conception of groups as sets is inadequate, they may begin to include the group operation in their determination of groups and subgroups. A student may realize, from appropriate experiences, that a given set will have a number of properties, one of which is that a binary operation satisfying certain conditions can be constructed and associated with the set. We saw some evidence that could be interpreted in this way.

We were intentionally vague about the operation when we asked if Z_3 is a subgroup of Z_6 in the written assessment because we wanted to see what operation the student chose (of at least two possibilities) and what role the operation played for her or him. When we ask for an explanation of this answer in the interview, it is the interpretation which the student then gives that forms the object of our interest. Consider, for example, the following exchange with Tim.

I: Is Z_3a subgroup of Z?

Tim: Ok, I'll say yes.

I: Ok, and how do you check that to be sure?

Tim: Ok, let me take that back. It depends on the operation.

6Z is the group of all integers with the operation of addition.

One interpretation of this exchange is that, for Tim, the set is the predominant as- pect of the group and the operation is secondary. But whether or not the emphasis of this conversation accurately represents Tim's complete understanding, it is apparent here that he at least exhibits some separation in these features of groups.

Groups as sets with operations

The group operation may be constructed on a set in a number of ways (a formula, a table, the operation induced by a group which contains the set). In general, the students we observed seemed to be quite comfortable with examples such as addition mod 3, group operation tables, and the operation of addition mod 6 induced on one or another of the subsets of Z_6. Making sense out of these various realizations of group operations requires that the student coordinate a number of general function concepts with ideas that arise in a study of groups.

Since an operation on a set is a function (both mathematically and psychogenetically) a student's function schema may intervene here with all of the power and difficulties related to the level of development of an individual's function schema (see Breidenbach et al, 1992 for a discussion of the psychogenesis of the function concept).

In any case, the students seemed to distinguish these operations individually and then organize them in some coherent manner. Initially the student may see that one operation is preferred (addition mod 3 in Z_3, composition of permutations inS_3, etc.) to others (e.g., the operation induced by Z_6 on {0, 1, 2}, or an arbitrary table.) At one point, Kim was discussing {0, 1,2} with the induced operation addition mod 6 because it is a subset of Z_6. In the middle of this discussion, she said the following. Kim: Ok, this is Z_3 I should be using addition mod 3.

A student may eventually encapsulate a set of objects and some operation on this set to form an object that is for her or him the conception of this particular group. After a de-encapsulation of the group object there may be other operations which could be applied to the set, but the one that came from the de-encapsulation may be preferred. For example, Lon's remark below could be interpreted this way. After some discussion regarding the operation induced by Z6 on the subsets {0, 1,2} and {0, 2,4} Lon says:

Lon: Z_3wants to have as its operation addition mod 3 but the {0,2,4} you mentioned is just fine and happy to have addition mod 6, to have it be a group.

<u>Subgroups as sets with operations</u>

As was the case with groups, the concept of subgroup may also become coordinated with an operation. At this point, for the student, a subgroup is a subset to which some operation has been attached, making it a group. Thus, in the following excerpt, it appears that for Z_3 to be a subgroup ofZ_6, Ann only requires that Z_3 is a group (under addition mod 3) and that its elements are all in Z_6

I: Is Z_3 a subgroup of Z_6?

Ann: Yes.

I: It is. Okay, do you want to write down the elements of z_g for instance.

Ann: Okay (writes, indicating the subset {0,1,2}.)

I: Okay, now you say that Z_3 is a subgroup ofZ_6?

Ann: Uh-huh.

I: And how do you see that?

Ann: Well, if you make the table for addition, its gonna be closed and it, uhm, gets all the properties.... (The interviewer probes to see which operation Ann is using.)

Ann: Modular arithmetic.

I: Right, which modular arithmetic?

Ann: Well, a3 (addition mod 3)

Cal puts it more succinctly in referring to Z_3 as a subroup ofZ_6.

Cal: Well, it's a subset of Z_6 and a group in itself.

The subgroup's operation is induced from the larger group

Once students appreciate the role of the operation, they are able to understand that the subgroup operation must be the same as in the larger group. This requirement may appear to the student to be somewhat arbitrary and it might not relate to the restriction of the operation as a function on the larger group to the subset. For example, in the following excerpt we cannot be sure that May realizes why the operations have to be the same, or even that she knows why addition mod 6 occurs.

I: IsZ_3 a subgroup ofZ_6?

May: Yes. Wait. (Pause) They're not even the same operation. It can't be.

I: Why not?

May: Because you'll get elements. Well, the definition says you have to be in the same operation, for one.

I: Ok, good point. If we're going to say that this is a subgroup, we're implying that its the same operation. What's wrong with the operation addition mod 6?

May: addition mod 6?

I: $Z_6$is the group.

May: We'd be looking at something like that. Z_3 is not in this group, that's the thing that keeps sticking in my mind they're not the same operation.

In this passage, May demonstrates a disequilibration that can occur when a conflict arises between the canonical operations on $Z_6$and Z_3 on one hand, and the "same operation" requirement of the subgroup concept on the other.

Some students were more explicit about where the operation comes from when a subset is being considered. In the following comment, Sam apparently understands that the operation on the subset is the restriction of the operation in Z_6.

Sam: Z_3 is the set {0,1,2} and Z_6 is the set {0,1,2,3,4,5}. To be a subgroup, it would have to be closed under whatever operation you are using inZ_6.

We suggest that a student's concept of function must intervene when he or she considers the operation o on the set G. Rather than consider the static situation in which the subset H of G has an operation which happens to agree with o, the student may perform the (mental) action of restricting the function to (pairs of) elements of H. At this point the issue of H together with o being a group is raised and there must be a coordination of the emerging subgroup concept with the student's group concept and function concept.

Coordination of Groups and Subgroups

As we have already begun to see, the individual's development of the subgroup concept may be coordinated with the development of the concept of a group. We can see some indication of students' understanding of group and subgroup when they are asked to determine whether a specific group is a subgroup of another group. While such a decision may once have been made considering the elements only, when a student understands the role of operations, a different approach is used. It is possible that the student would consider a subset to be a subgroup if it is closed under the induced operation.

For Sam, closure is apparently enough.

I: Now would you please look at this: {0, 2,4}, at this subset of Z_6. What can you say about it?

Sam: I don't know...It would be...a subgroup.

I: So why you're saying you don't know? You know. It is a subgroup of...?

Sam: Z_6.

I: ...of Z_6.

Sam: Because it would be closed under addition mod 6.

Ann also emphasizes closure, but she explicitly assumes that the group properties other than closure are inherited from the original group. This could be an example of awareness of the deficiency of relying on closure only and responding to disequilibrating experiences by looking for reasons to avoid reconstructions.

I: Suppose you look in Z_6 at the subset of elements consisting of 0, 2, and 4.

Ann: Okay.

I: Now that's a subset of Z_6, right?

Ann: Yeah.

I: Okay, uhm, would you say that subset is a subgroup?

Ann: Yes, I would that one.

I: Okay, I noticed you moved your fingers along each of the elements. What were you checking? What were you thinking about?

Ann: Closure.

I: You were thinking about closure.

Ann: Uh-hmmm.

I: Is closure all you need to be concerned with?

Ann: Well the other ones you told us were implied to the closure from Z_6.

Some students seemed to understand that in order for a subset to be a subgroup, all of the group properties, relative to the induced operation must be preserved. We observed Kim performing such a check as she determines that {0,2,4} is a subgroup of Z_6.

Kim: If I'm using mod 6 and I add 2 and 4, I'll get 0, if I add 4 and 4, I'll get 2, so its closed. O would be an identity. If I added 2 to 2 I would get 0-no, I would get 4 which is in the set so it is not the inverse. If I add 2 to 4, I'm going to get 0, so I'm going to have my inverse. So I'd say that's a group under addition mod 6, and therefore it's a subgroup.

She does not mention associativity at all. Lon, who does, may see that for the identity, it is only necessary that 0 be in the subset. But he may not yet have seen that the same is the case for inverses because he apparently checks, not that 4 is in the subset, which is enough, but that 4 is the inverse of 2, which is already known. Lon: Let's see. Associativity: inherited. Identity: yeah, 0. Inverse: 2 and 4 are inverses of each other, and 0 is its own inverse.

Understanding Groups as Objects

Encapsulating a process into an object can be extremely difficult for students. It may be much delayed or even not occur at all in some instances. Moreover, we do not know very much about how to help bring this about (see for example Sfard, 1992, but also Ayers et al, 1988 and Breidenbach et al, 1991.) Our point of view is that when a student is in a situation in which applying actions are required, then he or she may tend to encapsulate processes in order to have objects to which the actions can be applied. In other words, trying to treat something as an object can lead to making it an object.

Our notion of action includes the determination that a certain property is satisfied. Trying to perform such an action can help with the formation of an object. There must be a "something" that posseses the property and thinking about this may lead to encapsulation.

Another kind of action that requires understanding of groups as objects is to see that two groups (and their operations) may be the same, that is, isomorphic in a naive sense. An awareness of group properties can lead to a reconstruction of the group concept so that a group is seen as an object. This reconstruction can make it possible, in some contexts, to consider two groups in which the sets and operations can be informally matched to be the same.

Properties of groups

Group features such as the order of the group, being cyclic or commutative or being a group of symmetries are examples of properties. It is likely at this stage that Lagrange's theorem is applied in checking for a subgroup, actually, showing that a subset is not a subgroup. Of course there is considerable experience suggesting that it is very hard for students to reach this stage and many of them, apparently, do not. For example, some of our students, when asked to find a subgroup of order 4 ofD_3, stated explicitly, that such a subgroup doesn't exist since 4 does not divide 6. Others, however, tried to create various possible subsets of the elements of D_3 such as $\{R_0, a_1, a_2, a_3\}$ (the identity and the three flips), or {[1, 2, 3], [3, 2, 1], [3, 1, 2], [2, 1, 3]} (the identity and three other permutations), while still others simply listed set with four elements, not necessarily forming a group and not taken from D_3.

At this stage, also, when trying to determine whether one group is a subgroup of another, checking the group properties is simplified by the use of certain shortcuts. The student comes to realize and use the fact that of the four group properties, associativity is inherited, and that, in addition to closure, it is only necessary to check that the original identity is in the subset and that the inverse of every element of the subset is also in the subset.

Group as a generic object

The final step in the construction of the concept of a single group begins with the realization that other, apparently different, groups may be constructed, but they turn out not to be really different. At this point a developing (but still naive) conception of isomorphism may intervene and the student might construct the process of forming several specific groups and establishing isomorphisms between them (e.g., {0, 1, 2} with addition mod 3 and {0,2,4} with the operation induced by Z_6.). The encapsulation of this process would create an object which is the group in question.

None of the students that were interviewed made statements that could be interpreted as having taken this step. But we suspect that Lon is on the verge when he makes the following statement in reference to {0,2,4} which he knows to be a subgroup of Z_6.

Lon: As a matter of fact, it's Z_3, but, or, excuse me, it's isomorphic to Z_3.

Lon: The problem here is that you have two groups that are isomorphic to each other, and yet one is a subgroup of a certain group, and the other is not.

With a little more thought, Lon may soon be able to resolve the "problem" by identifying all of the isomorphic realizations of Z_3

Getting to Grips with Cosets and Normal Subgroups

This section kicks off by exploring how cosets are formed. We then look at how to try and define a new operation on these cosets, with the goal of creating a fresh group known as the quotient group, denoted G/H, from the original group G and subgroup H. However, this process doesn't always go smoothly – its success hinges on a property called normality, which a subgroup might or might not possess within the group.

It's widely recognized that the math behind these ideas can be really challenging for students, and the individuals in this study certainly weren't immune to that. While most students managed to build the cosets and work out their products in the fairly simple scenario involving the subgroup of all multiples of 3 within Z_{18}, things got tougher when they had to identify the quotient group as being equivalent to a more familiar group. Performance took a significant hit when they tackled similar questions involving D_3. This suggests that normality – a concept that really only comes into play with non-commutative groups like D_3– wasn't grasped well by many students.

As we'll explore further, the dip in performance was quite noticeable when the focus shifted from Z_{18} to D_3, and it became even more dramatic as the mathematical problems got more complex.

Understanding Coset Formation

In the following discussion we would like to suggest that an individual's concept. of formation of cosets could follow an action-process-object development which is dependent on the context.

The examples in this section are based on a generic group G, a subgroup H and the (left) cosets aH, a ∈ G where aH = {ah : h ∈ H}

We made use of the two examples:

G=Z_{18}, H=(3), and G = D_3, H = {I, R, R^2} where (3) is the set of multiples of 3 in Z_{18}, and I, R, R^2 represent rotation by 0, 120, and 240 degrees respectively. In Z18 we used a + H instead of aH for a coset. In discussing normality, we also considered right cosets.

<u>Coset formation as an action.</u>

Coset formation as an action is possible only in familiar situations and where explicit formulas are available.

For our students, the cosets of (3) in Z_{18} could usually be formed, but cosets in D_3 were more difficult. Often, students were be able to do the former but not the latter. For example, on Question 3 about Z_{18}, parts (a), (b), (c) were attempted by all but 3 students and part (d) was omitted by 4 students. On Question 4 about D_3,however, 8 participants omitted part (a) and 14 omitted part (b). (See Appendix I.) Many of the students explained their computation of cosets in Z_{18} much as Cal did in the example below. For Cal, this computation is probably no more than an action,since he seems to require explicit listing of the individual elements at each point.

Cal: Well, the number in front is what you add to each element inside the set. So zero added to these six elements would keep the same six. One [the number] added to each, which is in the first column, would give you the 1,4,7,10 and then you add 2 to these first the H which is 0 through 6,9,12,15. Then you add 2 to each and you get 2,5,8,11,14 and 17.

A possible misconception seen here is that the student may be confused about whether, in forming all cosets aH of a group G, a runs through each element of H or of G.

<u>Coset formation as a process</u>

The development of coset understanding from action to process may initially occur in familiar structures. As a student begins to interiorize cosets from actions to pro- cesses, familiar cosets such as those in Z_{18}may become processes, while for a complex coset, such as found in D_3, the student may have great difficulty even constructing an action.

At the transition, the gap between the student's ability to form cosets in $Z_{18}$and in D_3 widens. In the former case, the student who is thinking beyond formulas may begin to see patterns in sets such as

1+ (3) {1, 4, 7, 10, 13, 16}

There will be comments such as "The differences are all the same", "Every third one", "All that matters is where you start". In the following, Hal may be indicating an interiorization of the action by virtue of the pattern he expresses.

Hal: Z_{18} subgroup H which is generated by the element 3. Ok, I interpreted this as 0+ H,3+…… Every third element beginning with 0. So 1+ H every third element beginning with 1 in Z_1s and every third element beginning with 2. And that would generate all the elements that are in G.

We cannot be certain that Hal has really constructed a process for coset formation. In the written assessment, he was unable to do anything on Question 4 and, in the interview he required considerable prompting to construct the cosets in $D_3$of the normal subgroup {I, R, R^2}. It may be that coset formation is still an action for him but, with the aid of formulas available in Z_{18} he is apparently starting to see patterns as he moves towards construction of a process for coset formation. The lack of simple formulas to apply might explain the delay in the case ofD_3.

One thing to point out here is that understanding a simple concrete example, in this situation, does not seem to help much with extension to the general case.

Upon completion of interiorization, our theory suggests that constructing a coset would become a process that could be performed in a variety of situations. The student could think about doing it without actually making calculations. The student could not only form individual cosets, aH but could think of doing it for every a ∈ G. It is then necessary to decide that all cosets have been formed. The most obvious criterion is to stop when you start getting repetitions. Lon appears to be using such a method below:

I: When you were talking about the elements in these cosets and stuff, what happened to 3+ H and 4+ H?

Lon: Okay, yeah. I should have said that 3+ H, of course, which is a coset in its own right, is equal to the set 0+ H because you get the same members as if you had added 0. Same goes with 4,5,6 and so on.

Using some mathematical properties, some students will suggest shortcuts. This may be what Ann is doing in the following excerpt. She has just gone through the details of constructing the cosets in Z_{18} of the subgroup of multiples of 3. She seems to suggest that you stop making cosets when everything is used up and when you don't get new cosets.

Ann: Well, it took care of all the numbers and you start going when you put 3(plus 18)H, addition 18H, you're gonna get the same set as you got in 0.

Coset as an object

Thinking about cosets as actual objects can be a tricky step for students. Often, even as they start to grasp this idea, they still see a coset primarily as the result of a process—how it was made. A sign of this might be that they avoid giving cosets simple names (like 1+ H) and instead feel the need to list every single element within the set, or they might only use descriptive terms such as "the rotations" or "the flips."

In her written work, May calculated the cosets within Z18, listing each set and all its elements. After that, she gave them names like 0+ H, 1+ H, up to 17+ H, and showed that these all boil down to just three distinct ones: 0+ H, 1+ H, and 2+ H. However, when it came time to create the operation table for these cosets, she didn't use these labels; instead, she wrote out all the elements for every single set again.

Another hint that this transition isn't quite complete showed up during interviews. When students were asked to explain coset multiplication—something that requires viewing cosets as distinct objects—some found it hard and ended up explaining only how the cosets are formed initially. Talking about cosets in any way other than the process of creating them seemed difficult for them. Here's a typical example: An interviewer asked Ann to explain how she built the operation table for the cosets of (3) in Z_{18}. Before getting into the operations between the cosets, Ann felt she had to first explain the process she used to form them.

I: Can you explain to me how you got those uhm, uhm how you went about doing that, you know, setting up that table. How did you make it.

Ann: Well you just, if you take like 0 and add it to the sub-group, you get the same. elements of course. But if you add one to each one of those, you get that set {1,4,7} and so on. And if you add 2 to the H, you get those numbers there and you see that it encompasses all 18 numbers. So its like they're equivalent to like {1,2,3} err {0, 1,2}. I'd say mod 3, whatever you want to call and that's how I'd set it up.

When the transition to coset as object is completed, the student may give symbolic names to cosets and use these names in working with coset multiplication. Where necessary, the student can de-encapsulate the name back to the set as process and use that process in making calculations. Consider, for example, Lee's response when asked to explain how she computes the operation table. It is a clear example of de- encapsulating an object (set) into the process from which it came (the elements in the set).

Lee: First of all, my H+0 was the subgroup. And then H+1 would be the subset that contains {1,4,7}. And then 2 would be 2 uh, each element added to 2.

I: Okay.

Lee: So what I did initially when I set up my operation table is to say that those are possible subsets. Ok, when I operated on... Anytime I'm operating on this set, it remains that set. But those numbers will generate, in Z_{18} under addition, they will have those elements contained in them.

It should be noted that on the written assessment, Lee's operation table is written entirely in terms of labels for the cosets indeed, in all of Question 3, the only time she wrote a set was in specifying the identity in the quotient group $Z_{18}/(3)$, where she wrote e(G/H) = H = {0, 3, 6, 9, 12, 15}

We saw a similar set of responses from Lon. He and Lee were the only two students who succeed completely on question 4, regarding D_3.

Coset Multiplication

We see some parallels between the action-process-object development of coset formation which we just described and a similar construction for coset multiplication. Therefore we suggest that an individual's concept of coset multiplication may also follow an action-process-object development which is again dependent on the context. Indeed, even more than with coset formation, coset multiplication is particularly dependent on the domain. It seems possible to understand coset multiplication instrumentally in Z_{18} and be totally lost regarding the same operation in D_3.

In the written assessment, all but one of the students were able to compute the multiplication table for the cosets of (3) in Z_{18}, but for D_3, although 14 students were able to find a subgroup which was normal, only three of them worked out the products of cosets.

It is difficult to attribute any general understanding of coset multiplication to students who are very comfortable in one situation and completely lost in the other. Nevertheless, in the case of adding cosets of (3) in Z_{18}, we can discern what may be a development. There appears to be not only a movement from action to process for the method of adding two cosets by adding all elements in one to all elements in the other (which is how it was initially defined in the course), but also for the construction of the method of adding two cosets by adding representatives (which is, for our students, a theorem).

In the following examples, again the group is Z_{18} and H is the subgroup (3) of multiples of 3.

<u>Coset multiplication as an action</u>

One straightforward type of action in mathematics is computation according to a formula. Often, a student may succeed with computations even before the concept behind it is fully understood. This is apparently the case for coset operations. For example, it seems that for Tim in the comment below, the result of adding (the coset operation in this case) 1+ H and 2+ H is 3+ H just because this doesn't contradict his prior knowledge that 1+2 = 3.

Tim: [In explaining why the sum of 1+ H and 2+ H is H]

…..if you add what, 1+2, then this would be what, Z_3right? That would be 3 which in Z_3would be 0 and that is why I put 0.

It seems analogous to children's ability to compute correctly the sum of "one- seventh" and "two-sevenths" or "one barbarow" and "two barbarows" without understanding of what the manipulated objects are.

As the participants in this study started to develop addition of cosets as an action, there were several variations of the statement, "Add everything in a+H to everything in b+ H." For example, Ann explains it as an action in a specific example. She is asked how she determined that adding 1+ H to itself gives 2+ H.

Ann: By adding them together element by element, you see those elements are congruent to the ones in the 2+18 H, like I take 1+1 and you get 2, and then keep one going through the whole set, 1 plus 3 at 5, and so on...

Interiorization of the action to a process

In some cases, we can see what may be a movement towards constructing a process for coset multiplication that makes strong use of the specifics of addition in this case.

Lon: We have to define this operation which I suppose, for want of a better word, would be called set addition. We are adding all the members of the one set to all the members of the other set, again adding addition mod 18, and your result will be the set of all possible answers that you can get from a member of the first set plus a member of the second.

Construction of the method of adding cosets by adding representatives an encapsulation

Perhaps we can consider the method of adding two cosets by choosing a representative of each, adding the representatives and taking the coset of the sum as being essentially an encapsulation. That is, the process of adding all elements is encapsulated to the the single representative which is an object.

In the cases that we see here, the students all rely heavily on the specifics of the arithmetic inZ_{18}. They pick a very special representative and reason through the new method. Thus, the question of proving independence of representatives does not arise.

We can see this in Lon's continuation of his explanation.

Lon: So 1+ H plus 0+ H would give you - let's see, what's the easiest way to put this -every one of these members of 0+ H is a quote, unquote multiple of, yeah, I can say multiple of 3. Okay, every one of the numbers here is congruent to 1 mod 3, and when I add... To be in this set you have to be congruent to 0 mod 3. To be in this one, you have to be congruent to 1 mod 3. And when you add two numbers like that, you have to get a result that's congruent to 1 mod 3. And all those numbers. are the numbers in the set 1+ H, and no others.

De-encapsulation and addition by representatives

Some students showed an ability to go back and forth between the method of adding all elements of one set to all elements of the other and the method of adding representatives. This seems fairly clear in the following explanation by Gil. Asked to consider three examples of coset operation, he does the first by adding elements, the second by representatives, and the third by what could be a mixture of the two. It may be that his going back and forth only shows him in the midst of solidifying his encapsulation.

I: And what is the meaning of 1+ H the operation 1+ H gives you 2+ H?

Gil: This 1+ H is the coset of H that is arrived at by adding 1 to the elements of H. So if 1+H would be this one {1, 4, 7, 10, 13, 16}.

I: And now you have to make...

Gil: Right, and if I add that to this, it's like saying 1 plus every element in there, 4 plus every element in there, 7 plus every element in there, and so forth. And what I'm going to get as a result is a coset of H generated by 3 which is the same as {2,5,8, 11, 14, 17}. Does that make sense to you?

I: I am trying to understand. Would you please do it once again maybe take another element, 1+ H operation 2+ H.

Gil: Okay, the 1+ H and the 2+ H-H operated on itself is going to give me the H back so I don't need to worry about that. So the 1 plus the 2 is the same as 3, but 3-If I add a 3 to every element, I'd get the original set generated by 3. 0+3 is 3, 3+3 is 6, 6+3 is 9, on down the line, 15+3 is 18 which is 0 in Z_{18}

I: And how do you add 0? For example, if we had 2+H+0+...-excuse me -2+ H operation with 0+H?

Gil: Okay, that would be taking one of the elements, or any element here and adding it to every element there, so if I take 2+ the 0, I am going to get a 2. And if I add 5+01 am going to or, excuse me, 2+0 is 2, 5+0 is 5, 8+0 is 8, 11 +0 is 11, so I am re-identifying that same set,...

Most of the students interviewed made much less progress with cosets in D_3. In the case of Z_{18}, some students seemed to rely on the fact that 1 is first in {1, 4, 7, 10, 13, 16} and 0 is first in {0, 3, 6, 9, 12, 15} so that the addition of these two cosets can be ob- tained by adding the two first elements, 1+0=1 and taking the coset which begins with 1. They had no such mechanical device in D3 and several tried to find one. In general, they were not successful and the little progress they made required a consid- erable amount of prompting.

Constructing the Concept of Normality

Our students did not appear to have constructed very much of a viable coset concept to use in dealing with normality. Therefore we have only a few examples to illustrate a development which is mainly derived from the general theory with which we are working and our own understanding of normality. This development, we conjecture, begins with the formation of a left coset aH discussed above and extending this process to the analogous construction of a right coset Ha. Thereafter the student would be able to coordinate the two processes aH, Ha with equality to obtain aH = Ha. Then the process aH = Ha can be encapsulated into an object which has a boolean value (true or false) that can vary with the element a of G. Thus the student could construct a function which assigns to each a ∈ G the truth value of the assertion aH = Ha. This function can then be iterated over its domain and universal quantification can be applied to obtain a single value, true or false. Finally, we would expect the student to thematize all of this into a schema which can be applied to any situation involving a group and a subgroup.

The examples we do have illustrate what may be an iteration of the boolean valued function over G and the formation of the schema for general application.

Constructing aH, Ha, and verifying their equality

We take the following description by Gil as an indication of the construction of the equality for a particular choice of a. He knows about doing it for different values of a, but he makes the error (repeated by several students) of using only the elements of H.

I: Okay, why do you think it is normal?

Gil: Because it...

I: Just to make you relax, yes it is normal.

Gil: It is a subgroup of G and no matter what element of that subgroup I apply to the elements for example, if I choose 3 and this will work for every element, if I take 3 and I operate on H, that is the same as taking H and operating on 3 so it is kind of like a commutative type thing. If this is a normal subgroup, the best I understand it, any element a of H, operated on H is the same as H operated on a.

Iterating the boolean valued function a I→ (aH = Ha) over all a in G

To confirm the normality, a student learns to check if the equality (aH = Ha) holds over all a in G.

Since Kim had written the statement aH = Ha in her response to Question 4(a) on the written assessment, the interviewer asked her to explain.

I: What is, well H is clear because the problem says that is a subgroup. What is a?

Kim: a is an element in D_3.

I: An element inD_3, good. One particular element in D_3?

Kim: No, I don't think so.

I: No, what, which element? Because it doesn't say here, it just says aH = Ha.

Kim: It could be any element. If any element does that, it is going to be normal.

Thus, in her last statement, Kim may be expressing a boolean valued function whose domain is D_3("any element") and value is the truth or falsity of the assertion aH Ha ("does that").

<u>Thematizing the previous step into a schema that can be applied in any specific situation.</u>

Gil states this clearly in describing how he checked that a subgroup is normal.

Gil: To check if this is a normal subgroup, I operate this on that element and produce this (aH). What I am going to do now is to reorder these, and put this one first which means I am going to do this part of the composition first (Ha). So, and what I am doing is checking to see if that answer is the same as that answer (aH = Ha), and if this is true in all cases, then this would be a normal subgroup.

Note that he may or may not still think that "all cases" refers to H and not G.) As with coset multiplication, there was confusion here in passing from normality of subgroups in Z_1s to normality inD_3. In the former case, students were generally able to begin applying the definition and saw fairly quickly that it would always work because Z_{18} is commutative. In passing to D_3, however, the schema broke down completely for most students.

Interestingly, there is one example in which it went the other way. Hal had difficulty in checking the normality of (3) in Z_{18}, but eventually succeeded. Shortly thereafter, he was able to handle normality in D_3without difficulty. Although, for most students, success with Z_{18}did not carry over to D_3, Hal seemed to be constructing sufficient understanding of normality by working through Z_{18} in the interview. He seemed able to subsequently extend the concept to D_3 as we talked to him.

Interpretations

In this part, we're trying to summarize our understanding of what's likely happening as students attempt to grasp group concepts. It's important to note that we only consider the statements here to be plausible explanations based on the data we've collected. These ideas were shaped by our theoretical viewpoint and our own familiarity with the math involved. Ultimately, our goal in drawing any conclusions is to suggest ways teachers might adjust their methods to help students learn about groups more effectively.

From observing this group of students, it seems like the concepts of a “group” and a “subgroup” tend to develop somewhat in parallel. While it might seem like each concept could follow a fairly straight path of development, they actually seem to grow together quite a bit. Often, making progress in understanding one concept seems to depend on or wait for developments in the other. We’ve seen many specific examples where students use their understanding of one idea to build new insights about the other.

Furthermore, it appears that students learning about groups and subgroups also need to draw on other mathematical concepts they already know. Two of the most crucial ones for this process seem to be the concept of a “set” and the concept of a “function.”

It's also possible that the way these understandings unfold could depend on the order in which students first encounter these different concepts and how they specifically interact with each one.

Groups, Sets, and Functions

It seems that the idea of a group starts out very basic, tied closely to how a student first thinks about a set. As the student learns more, they start to notice various properties that a set might have. One of these properties is a binary operation, but initially, there might not be a clear distinction between these different properties.

An Important moment in this learning process is when the student starts to pay special attention to the binary operation and really focuses on its role as a function. For this to work mathematically, the student needs to have a good grasp of functions that take two inputs.

It looks like the final step In this process is combining two ideas—the set and the function (which is the binary operation)—into a single pair. This might be the student’s first real grasp of what a group is. Later on, this understanding needs to be refined to a higher level, incorporating the idea of isomorphism. Then, the student can view a group as a collection of pairs that are essentially the same in structure (isomorphic).

In the course described, the participants began working with binary operations right away. The focus was initially on the operation itself rather than the set it acted upon. The set (the domain) was introduced more formally later and was discussed alongside the concept of functions with two variables.

Groups and Subgroups

I think the connection between a group and its subgroups comes down to the idea of a function. After going through a few stages of learning, a student's understanding of a subgroup might really solidify around the concept of restricting a function to just a part of its domain.

It seems like there's a strong similarity in how we learn about groups and subgroups. For instance, when a student first sees a group as just a set with a bunch of properties, they might view a subgroup as simply a smaller set where all the elements share one specific property or meet a certain condition. Paying attention to the binary operation as a function seems to be key for grasping both ideas.

Just like with groups, people first learned about subgroups by using computers to build examples and write programs to check properties. The formal definition they used was the usual one, and it didn't specifically call out restricting a two-variable function to a subset.

<u>Understanding Complex Ideas</u>

It's evident from our interview notes that how people grasp basic concepts related to mathematical groups is pretty intricate. You see all sorts of interpretations, mistakes, and misunderstandings, plus the trouble students have moving from modular groups to permutation groups. Also, the way these ideas develop in someone's mind isn't a straight line—it's all over the place. All these student reactions really show that figuring out even the very start of abstract algebra is a big deal in how math students' minds grow. This is especially true since, more often than not, abstract algebra is one of the first classes students take that isn't just about memorizing formulas and copying solutions to standard problems.

This complexity shouldn't be surprising when you consider what other researchers have found studying less complex math topics and even areas totally unrelated to math.

For instance, Schoenfeld and his team in 1990, along with Kaput in 1992, demonstrated just how complicated even simple ideas like slope and linear functions can be in a student's mind. Piaget, back in 1977, spent a lot of time studying how these specific concepts develop in people, and he found that the process is long, tough, and often full of confusion and isn't a smooth path. We should definitely expect things to get even more complicated as the math gets more advanced.

Understanding Student Mistakes and Misunderstandings

Experts in math education, like Ferrini-Mundy and Gaudard in 1992, have pointed out that the mistakes students make can offer a peek into how their minds work as they learn. Several learning theories try to explain errors. For instance, the idea of "epistemological obstacles," introduced by Bachelard in 1938 (and discussed further by Sierpinska in 1992), suggests these errors might be stages learners need to go through. Instead of just trying to get rid of them, we can help students avoid getting stuck at these stages.

Similarly, Piaget's general theory from 1975 includes the idea that concepts are built at a level suitable for a learner's current math experiences. However, when new things are introduced, concepts need to be rebuilt at a higher level. So, if this rebuilding is delayed, a student's understanding of a math idea might work at one level but be wrong at another.

We tend to agree with this more positive view of errors. Our own observations support the idea that errors and misunderstandings can guide us toward helpful and accepting responses that improve learning. For example, when students are learning about subgroups, it's key for them to compare the elements of two groups and check if all the elements of one group are also in the other.

Well, it's really no surprise that some students might jump to the conclusion that one group is a subgroup of another just because they see the elements of one group inside the other. We think of this "mistake" as a natural part of the learning journey. The key isn't to correct the student's thinking, but to help them move forward by considering not just the set membership but also the group operation itself.

More generally, consider the developmental steps that we have used to describe how various topics are being learned. On the one hand, it seems entirely reasonable to pass through these sequences. On the other hand, a student who happens to be at a particular stage (other than the "last") may seem to have a misconception if he or she is trying to deal with a situation that requires construction of the next step. Such a situation may be quite healthy and productive. Again, what is to be avoided is a student getting stopped at an intermediate developmental stage. This certainly appears to happen, and when it does, it really is a difficulty that needs to be overcome.

Teaching Tips

For many students, their early math journey is all about mastering algorithms to tackle similar problems over and over. Abstract algebra brings a sudden shift—from focusing on methods to grasping concepts and understanding the bigger picture. This change makes it clear that teaching this subject the old-fashioned way probably won't work well. In fact, it might be that abstract algebra isn't doing very well at many colleges, even though there's no official research, lots of stories seem to suggest this is true. So, it looks like we really need to think about different, more creative ways to teach.

In this last part, we'll think about teaching methods based on what we've seen in this paper and other related studies. Our current work isn't advanced enough for us to offer much more than a few general ideas right now. Future studies and their findings will need to explore how to actually create and carry out these teaching methods, and also figure out how well they work.

When we think about how concepts are learned, the next question is usually about how to structure teaching to match that learning process. We want to be clear: we're not saying the concepts in this paper should be taught one after another in a strict, step-by-step order, just like the stages of development we've described. The reason our explanation follows a linear path is mainly for clarity in this written document. However, we do believe that the general progression we've outlined holds true—meaning, all the steps are involved, and you can generally see a similar order. But the actual journey for learners isn't a straight line. Research should aim to find ways to teach in a way that accommodates this natural, non-linear learning process, and we discuss this further in the section called "Finding Alternatives to Linear Sequencing."

For now, in this part, we're focusing on the individual steps themselves. How can we guide students to make a specific leap in understanding a concept? More specifically, what techniques can we use to help students really internalize actions, build processes, reverse processes, connect different processes, create new processes, wrap their minds around processes as single entities (encapsulation), construct objects, and finally, bring all these processes and objects together into larger frameworks or schemas?

In many places, Piaget has emphasized that the major constructions in cognitive development cannot be accelerated by education (Piaget, 1972), but it is possible to enhance the experiential base of students to enrich the developments that do take place (Inhelder et al, 1974). Sfard, 1992 has discussed the exceptional difficulty of getting students to construct objects out of processes and even suggests the possibility that what we are calling encapsulation may be beyond the ability of some students

In the case of thematizing a collection of processes and objects into a general schema, we feel that an essential requirement is that students reflect on the actions they are performing. Working together in cooperative teams, especially when students are talking about how to implement specific mathematical concepts on the computer can provide opportunities for discussion and reflection.

Let's talk about misunderstandings and getting stuck. How can we help students avoid the kinds of mistakes mentioned in this paper, or at least help them realize when they've made them?

We believe that the best way to fix a misunderstanding is to create a kind of "upsetting" contradiction between what a student thinks they know and what they experience. But there's a catch: because people tend to interpret their experiences based on their current knowledge—including any misunderstandings—they might seem to ignore these contradictions.

Once again, we recommend working in teams and using computers. It seems students are more likely to seriously think about contradictions pointed out by their peers rather than just accepting them from their teachers (Vidakovic, 1993). When it comes to computers, they appear to have a much greater ability to shake up a student's way of thinking compared to any human.

As for the last point, we can share a personal story from an elementary discrete mathematics course. Students often struggle to accept that the boolean value of P Q is true when both P and Q are false. Our experience using computers in Discrete Mathematics (Baxter et al, 1988) shows that when students enter such an expression into a computer and it returns "true," they are more likely to go through a learning process where their understanding is challenged and then rebuilt, rather than just having this idea presented in class.

Meeting prequisites.

We feel strongly that how students think about sets and functions really matters when it comes to grasping the group ideas we're exploring. We believe there are some key things students need to get comfortable with. They need to understand sets not just as actions of grouping things together, but also as distinct objects that can be part of other sets and be affected by functions. Students should see functions primarily as processes, especially focusing on how a function's action can be limited to just a part of its starting set. This helps them build the idea of a group acting on a smaller set. They also need to really grasp properties of functions, like one-to-one and onto, to work effectively with permutations. And finally, to get into specific group examples like permutation groups and automorphism groups, students must be able to think of a function itself as a single, solid object, not just a process.

<u>Finding Alternatives to Linear Sequencing</u>

Many years ago, Jerome Bruner proposed, for mathematics curricula, that a spi- ral curriculum should replace the simple, but simplistic linear sequence in which a complex topic is broken up into a logically coherent sequence of small units. Starting with the proposition that "any topic can be taught in an intellectually honest manner, at any age", Bruner suggested that full-blown mathematical concepts be taught at very early ages, albeit in naïve forms. Then, in subsequent years, the topics should be revisited and considered repeatedly at successively higher levels of sophistication (Bruner, 1964). This philosophy was one of the driving principles of the "new math" curriculum.

We believe very strongly that an alternative to linear sequencing must be found. Many students do come to abstract algebra courses without strong conceptions of functions and some are even ill-equipped to deal with sets. Perhaps abstract algebra is where they are supposed to strengthen their conceptions of these topics. Just as generals seem to always plan to fight the previous war, students appear to always be learning the material of the previous course. It may not be so bad if we can arrange matters so that a student is no worse than "off by one", throughout her or his schoolLife.

We feel, however, that evidence of how people learn mathematics from this and other studies suggests that Bruner's spiral curriculum may not be the last word on the subject. For example, in all of the various excerpts we have given from the student Kim, one can see that this individual appears to be, during a very short period (just a few days at most) at several different levels of development simultaneously. Neither a linear nor spiral sequence may be appropriate for such a student.

One alternative that has been suggested is called learning by successive refinement, a method common in software development (Wirth, 1971). In this approach, the student is always dealing with the whole except that he or she goes through a sequence of simplified versions of it, each one a bit more complex and more like the final version than the previous one. The successive refinements can take place along various dimensions such as degree of formality, language, global vs. local, or degree of generality. (See Leron, 1987.)

Another possibility is the microworld which is a software environment that establishes its own complete and consistent environment that represents a non-trivial piece of knowledge and is attractive to the intended audience. (See Papert, 1980.) The use of microworlds does not prescribe the interaction with the learner, but is built into the constraints of the system. The basic metaphor is that children learn by living in this world much as the children observed by Piaget were learning by interacting with the “real” world. They build mental models, make conjectures, try them out, and refine them when they are found wanting.

One might even consider a curricular point of view that is different from all of these and may be seen as something like an holistic spray. That is, using computers, and anything else available, students might be thrust into an environment which contains as much as possible about the group concept. The idea is that everything is sprayed at them in an holistic manner. Each individual (or team) tries to make sense out of the situation that is, they try to do the problems that the teacher asks them to solve, or to answer the questions which the teacher or fellow students ask. In this way the students enhance their understanding of one or another concept bit by bit.

They keep coming at it, always trying to make more sense, always learning a little more, and sometimes feeling a great deal of frustration. And it is the role of the teacher, not to eliminate this frustration, but to help students learn to manage it, and use it as a hammer to smash their own ignorance.

It could be that such a curricular design may be more in consonant with how people can learn mathematics. It seems that the data and interpretations in this paper are not inconsistent with such a view. In the future, we hope to report on teaching experiments based on such a philosophy so that we may be able to see if this is a direction in which learning mathematical concepts can be improved.

Conclusion

In conclusion, the core idea of group theory is a vital part of abstract algebra, with uses that stretch far and wide across many areas of math and science. In this study, we delved into the key definitions and traits of groups, such as closure, associativity, identity, and inverses. We also looked at important types of groups, including cyclic groups, permutation groups, and symmetric groups, along with essential concepts like subgroups, cosets, and homomorphisms.

Group theory gives us a formal way to grasp symmetry and structure. It makes complicated math problems easier to tackle by sorting and examining objects based on their algebraic behavior. This makes it especially handy in fields like physics (think quantum mechanics and crystallography), chemistry (molecular symmetry), and computer science (cryptography and coding theory).

Besides that, the abstract nature of group theory sharpens mathematical thinking and problem-solving abilities, helping students and researchers build a solid base for more advanced subjects like ring theory, field theory, and representation theory. Grasping these fundamental concepts not only boosts mathematical knowledge but also unlocks a wealth of both theoretical and practical uses. So, studying group theory is both intellectually rewarding and practically important in the wider world of modern mathematics.

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